ANALYTICAL SOLUTION OF TRANSIENT TEMPERATURE IN CONTINUOUS WAVE END-PUMPED LASER SLAB
Reduction of Temperature Distribution and Time of Thermal Response

by

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An analytical solution of transient 3-D heat equation based on integral transform method is derived. The result are compared with numerical solution, and good agreements are obtained. Minimization of response time and temperature distribution through a laser slab are tested. It is found that the increasing in the lateral convection heat transfer coefficient can significantly reduce the response time and the temperature distribution while no effect on response time is observed when changing pumping profile from Gaussian to top hat beam in spite of the latter reduce the temperature distribution, also it is found that dividing the pumping power between two slab ends might reduce the temperature distribution and it has no effect on thermal response time.

Key words: *integral transform method, slab lasers, end-pumping, thermal response, temperature distribution*

Introduction

High average power solid-state lasers suffer from the limitation of the possible heat that could be dissipated from the active element (AE). Slab shapes give an advantage of increasing the surface at which heat can flow out of the AE which increase the amount of dissipated heat that could be extracted from the AE thus increasing the possible output power by increasing the pumping power [1-3]. The maximum incident pumping power is limited by the stress fracture, which is caused by non-uniform temperature distributions in the crystals with pump loading [4], also reducing temperature distribution resulting from increasing the dissipated heat permits a significant enhancement in beam performance.

With the availability of increasing output power in laser systems as they use slab crystal, the study of their temperatures and the ability of reaching a quickly steady-situation attracts much attention. The ease of high cooling rate that could be achieved on slab laser with high induced power in satisfying the increasing demands on a high laser power system hindered the problem of reaching failure stress during operation, which may subsequently lead to medium break.

Usually, numerical method used to solve this problem, which is a time consuming technique also it requires long exercising. To avoid these limitations, analytical solutions are preferred in solving these problems since they usually consume less time, having an explicit form so one can indicates the effect of various factors in the result with clear physical mean-

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ing [5-7]. Few literature dealt with analytical solution of transient temperature in laser slab. Some of them are that presented by Sabaeian [8] who presented analytical solution for anisotropic transient heat equation for solid-state laser crystal with cubic geometry with robin boundary condition using separation of variable technique. Another work presented by Zhang *et al.* [9] solved the heat equation in 3-D model for anisotropic crystal with special boundary condition.

In this work, an analytical solution of transient 3-D heat equation based on integral transform method is derived, with the most possible boundary conditions and types of heat generations that could be imposed. The result is compared with numerical solutions presented by other works and good agreements are obtained, which verify the presented theory and the derived equation.

Theory

A slab crystal is studied which is pumped from its end to induce the required population inversion that generate laser which is usually combined with heat generation within the slab. The study of thermal effect on laser slab could be achieved by obtaining its temperature distribution which could be obtained by solving the transient 3-D heat equation, fig. 1(a) and (b), then [10]:

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} + \dot{Q} = \rho c \frac{\partial T}{\partial t} \tag{1}$$

Here k_x, k_y, k_z [$\mathrm{Wm^{-1}K^{-1}}$] are the thermal conductivities for x-, y-, z-directions, respectively, ρ [$\mathrm{kgm^{-3}}$] – the mass density, c [$\mathrm{Jkg^{-1}K^{-1}}$] – the specific heat, $\dot{Q}$ [$\mathrm{Wm^{-3}}$] – the heat source density, and T [K] – the temperature.

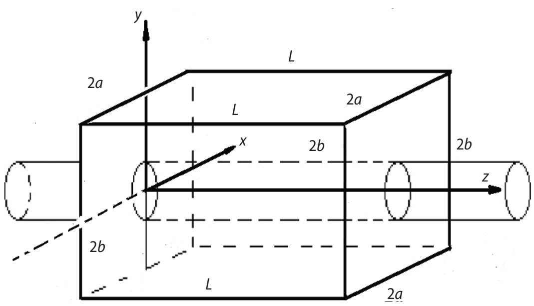

Figure 1(a). Laser slab geometry and pumping location

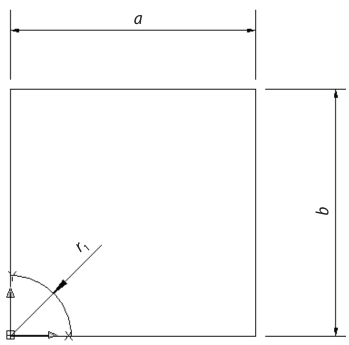

Figure 1(b). First quadrant for the face of slab where *a* and *b* is the transverse dimensions of the salb, r_1 = *a* for top hat beam and $r_1 = 2w_p$ for Gaussian beam

This equation can be solved by the successive application of integral transform to the x, y, and z variables, The part of absorbed power that converts to heat act as a heat source and it can has different profile [5, 6, 11].

Gaussian transverse profile

– For one end-pumping:

$$\dot{Q}(x, y, z, t) = \frac{2\eta\alpha P_{ab}}{\pi w_p^2} \exp\left[-\frac{2(x^2 + y^2)}{w_p^2} - \alpha z\right] \tag{2}$$

where η is the thermal factor, α – the absorption coefficient, P_{ab} – the absorption power, and w_p – the waist radius of pumping beam.

– For dual end-pumping:

$$\dot{Q}(x,y,z,t)=\frac{2\eta\alpha P_{ab}}{\pi w_p^2}\exp\left[-\frac{2(x^2+y^2)}{w_p^2}\right]\{\exp(-\alpha z)+\exp[-\alpha(l-z)]\} \tag{3}$$

where l is the length of laser slab and x and y [m] are co-ordinates.

Top-hat transverse profile

– For one end-pumping:

$$\dot{Q}(x,y,z,t)=\frac{\eta\alpha P_{ab}}{\pi a^2}\exp(-\alpha z) \tag{4}$$

where a is the radius of top-hat beam.

– For dual end-pumping:

$$\dot{Q}(x,y,z,t)=\frac{\eta\alpha P_{ab}}{\pi a^2}\{\exp(-\alpha z)+\exp[-\alpha(l-z)]\} \tag{5}$$

the last term in eqs. (3) and (5) represent the additional pumping source from the second end.

The basic steps in the solution of the problem can be summarized:

(1) develop the appropriate integral transform equations,
(2) remove the partial derivative from the differential equation, and
(3) solve the resulting ordinary for the transformation of temperature subjected to the transform initial condition.

Then the general solution for 3-D domain can be written [10]:

$$\theta(x,y,z,t)=T(x,y,z,t)-T_\infty=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{X(\beta_m,x)Y(\gamma_n,y)Z(\eta_p,z)}{N(\beta_m)N(\gamma_n)N(\eta_p)}\cdot$$

$$\cdot\exp\left[-\frac{k}{\rho c}\left(\beta_m^2+\gamma_n^2+\eta_p^2\right)t\right]\left\{\frac{1}{\rho c}\int_{t'=0}^{t}\exp\left[\frac{k}{\rho c}\left(\beta_m^2+\gamma_n^2+\eta_p^2\right)\right]\bar{\bar{g}}\left(\beta_m,\gamma_n,\eta_p,t'\right)\mathrm{d}t'\right\} \tag{6}$$

Here T_∞ is the ambient temperature which is usually taken to be equal to 25 ºC which is also equal to the initial temperature distribution through the slab. The $N(\beta_m)$, $N(\gamma_n)$, and $N(\eta_p)$ are the norms which are the square integral of eigenfunctions X, Y, Z at roots β_m, γ_n, and η_p of the equations, respectively. The triple transform is defined:

$$\bar{\bar{g}}(\beta_m,\gamma_n,\eta_p,t')=\int_{x'=0}^{a}\int_{y'=0}^{b}\int_{z'=0}^{L}X(\beta_m,x')Y(\gamma_n,y')Z(\eta_p,z')\,g(x',y',z',t')\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \tag{7}$$

The eigenfunctions, the normalization integrals and the eigenvalues are obtained depending on the boundary conditions. Due to symmetrical nature of the problem then only first quadrant of the slab is account for where the boundary conditions are:

– the front and back surface of laser slab are commonly both exposed to stagnation ambient air (*i. e.* at $z = 0$ and $z = l$) where l is the length of the slab then:

$$Z(\eta_p,z)=\eta_p\cos(\eta_p z)+H_a\sin(\eta_p z) \tag{8}$$

where $H_a = h_a/k$ and h_a is the convection heat transfer coefficient at the ends surfaces of the slab equal to 27.5 W/m^{2}K and its Norm is:

$$N(\eta_p) = 0.5\left[(\eta_p^2 + H_a{}^2)(l + \frac{H_a}{\eta_p^2 + H_a{}^2}) + H_a\right] = 0.5\left[(\eta_p^2 + H_a{}^2)l + 2H_a\right] \tag{9}$$

while the eigenvalues are the positive roots of:

$$\tan\eta_p l = \frac{2\beta_m H_a}{\eta_p^2 - H_a{}^2} \tag{10}$$

– convection boundary conditions are assumed at the outer lateral slab surfaces. Since only one quadrant is studied due to symmetrical nature of the problem then insulated boundary conditions are assumed at zero x- and y-axis co-ordinates (*i. e.* symmetrical pumping and symmetrical boundary conditions). These cases are generally acceptable boundary conditions in the slab geometry that usually used as laser medium. For insulated inside surfaces and convection boundary conditions at the outer surfaces, the eigenfunction can be written:

$$Y(\gamma_n, z) = \cos\gamma_n y \quad \text{and} \quad X(\beta_m, z) = \cos\beta_m x \tag{11}$$

The Norms are:

$$N(\beta_m) = 0.5\left[\frac{a(\beta_m^2 + H_c{}^2) + H_c}{\beta_m^2 + H_c{}^2}\right] \tag{12a}$$

$$N(\gamma_n) = 0.5\left[\frac{b(\gamma_n^2 + H_c{}^2) + H_c}{\gamma_n^2 + H_c{}^2}\right] \tag{12b}$$

where $H_c = h_c/k$ and h_c is the coefficient of convection heat transfer from the lateral surfaces of the slab assume its value is 20000 W/m^{2}K unless other mentioned.

While the eigenvalues are the positive roots of:

$$\beta_m \tan\beta_m a = H_c \tag{13}$$

and

$$\gamma_n \tan\gamma_n b = H_c \tag{14}$$

It is assumed that the Rayleigh length is much greater than the length of the laser slab so that the pumping radius is assumed to be approximately constant through the laser slab then the transform function *g* in eq. (7) can be written for the two types of pumping beam profile:
(1) for circular Gaussian beam distribution:

$$g_{mnp}(\tau) = \int_{x=0}^{2w_p}\int_{y=0}^{2w_p}\int_{0}^{t} \frac{2\eta\alpha P_{ab}}{\pi w_p^2}\exp\left(-\frac{2r^2}{w_p^2} - \alpha z\right)\cos(\beta_m x)\cos(\gamma_n y)\cdot$$
$$\cdot\left[\eta_p\cos(\eta_p z) + H_a\sin(\eta_p z)\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{15}$$

where $r^2 = x^2 + y^2$. Assume the pumping radius is equal to twice the waist radius of the pumping beam so that 99.9% of the pumping power is included.

(2) for circular top hat pumping beam:

$$g_{mnp}(\tau) = \int_{x=0}^{r_1} \int_{y=0}^{r_1} \int_{0}^{t} \frac{2\eta\alpha P_{ab}}{\pi r_1^2} \exp(-\alpha z)\cos(\beta_m x)\cos(\gamma_n y) \cdot \\ \cdot\left[\eta_p \cos(\eta_p z) + H_a \sin(\eta_p z)\right] \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{16}$$

where r_1 is the radius of top hat pumping beam and is equal to $2w_p$.

These integrals can be carried out numerically to obtain the final value of the functions.

Replacing the form of Norms, and carrying the time integration, the final form of eq. (6) can be written (for convection at the sides of slab and at slab ends):

$$\theta = T - T_\infty = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{4\cos(\beta_m x)\cos(\gamma_n y)\left[\eta_p \cos(\eta_p z) + H_a \sin(\eta_p z)\right]\left(\beta_m^2 + H_c^{\,2}\right)\left(\gamma_n^2 + H_c\right)}{\left[a\left(\beta_m^2 + H_c^{\,2}\right) + H_c\right]\left[b\left(\gamma_n^2 + H_c^{\,2}\right) + H_c\right] k\left(\beta_m^2 + \gamma_n^2 + \eta_p^2\right)} \cdot \\ \cdot\left\{1 - \exp\left[-\frac{k}{\rho c}\left(\beta_m^2 + \gamma_n^2 + \eta_p^2\right)t\right]\right\} g_{mnp}(t) \tag{17}$$

Even this form is derived for laser slab that has convection boundary conditions from its six facets. It can be used for lateral surfaces of the slab that have zero temperature difference between the surface and ambient temperature by increasing convection heat transfer coefficient. Insulated front and back surfaces of laser slab can be simulated by reducing the convection heat transfer coefficient to the stagnant air much beyond than that of the naturally cooled surface (*i. e.* much less than 27.5 W/m^{2}K.)

Validation

The steady-state maximum temperature obtained from this work is compared with that obtained from [10] where different pumping power are tested for the same slab and boundary conditions including the circular Gaussian beam with two different waist beam (0.4 and 0.225 mm). The percentage of maximum difference between the results between numerical solution for unsaturated condition and the analytical solution of this work were found to be equal to 3.8%., (see tab. I and II in reference [9]).

Also, the result of this work is compared with numerical solution for the temperature distribution in an Nd:YAG conventional slab where a laser slab having 3 × 3 × 5 mm^3. The thermal and optical properties of Nd:YAG were taken from [12]. The slab is end pumped by Gaussian beam having waist beam radius of 0.225 mm, assume the convection boundary conditions is as that mentioned earlier (*i. e.* h_a = 27.5 W/m^{2}K, h_c = 20000 W/m^{2}K), the heat load (*i. e.* ηP_{ab}) was 5 W, a very nearby result was obtained which verified the steady-state temperature distribution solution, fig. 2.

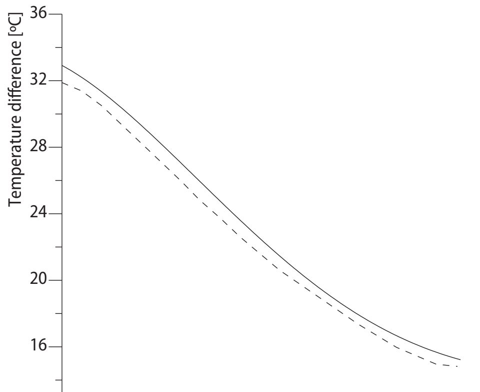

Figure 2. Temperature distribution through longitudinal axis of the slab, solid line indicates analytical solution of this work, dashed line indicated numerical solution as obtained by [12]

Result and discussion

In this work, the analytical solution of transient temperature distribution through the laser slab that end pumped by either Gaussian or top hat beam is derived based on integral transform method. The effect of different boundary conditions, type of pumping (top-hat or Gaussian beam), and one or dual end pumping are studied to predict how a rapid response can be achieved and how to reduce the temperature distributions through the laser slab. Replacing the values of Norms, obtaining the values of the roots together with slab dimensions of $3 \times 3 \times 5$ mm^3 which has a convection boundary conditions of h_a = 27.5 W/m^{2}K and h_c = 20000 W/m^{2}K. The thermal and optical properties of Nd:YAG is taken from [11], then the transient temperature distribution can be obtained using eq. (17).

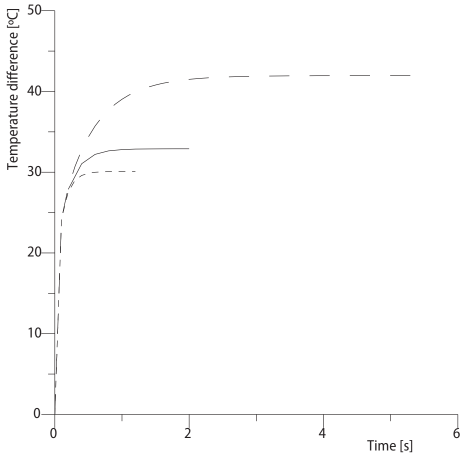

Figure 3. Temperature history of maximum temperature is laser slab at *x*, *y*, *z* = 0; solid line for h_c = 20000 W/m^{2}K, long dashed line for h_c = 5000 W/m^{2}K, and small dashed line for h_c = 100000 W/m^{2}K

It is found that increasing convection heat transfer coefficient can speed the thermal response, this is explainable since more rate of energy will transfer out of the slab which reduces the time required to achieve thermal equilibrium (named as time of thermal response), fig. 3. Note that this time is equal to the time where the temperature everywhere in the domain is equal to 99% of its steady-state value). Also the temperature distribution (including the maximum temperature that could be reached in the slab) is reduced as the convection heat transfer coefficient increase, figs. 3-5.

Changing pumping profile from Gaussian to top-hat beam leads to decrease in the temperature distribution that could be reached in laser slab assuming constant convection heat transfer coefficient and environmental temperature, figs. 4 and 6. It is observed that changing beam profile has insignificant effect on the time at which

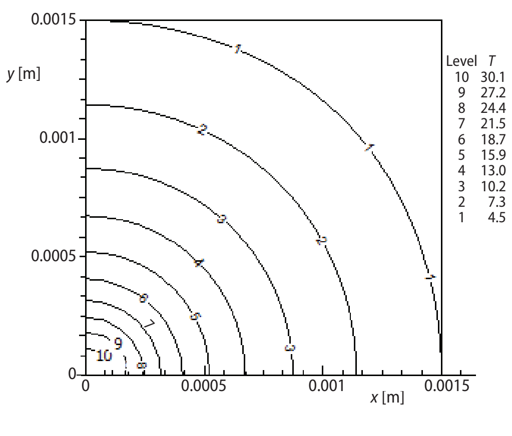

Figure 4. Temperature distribution at steady-state situation at *z* = 0 where the maximum temperature is 32.9 ºC, the slab is one end pumped by Gaussian beam, h_c = 20000 W/m^{2}K, h_a = 27.5 W/m^{2}K

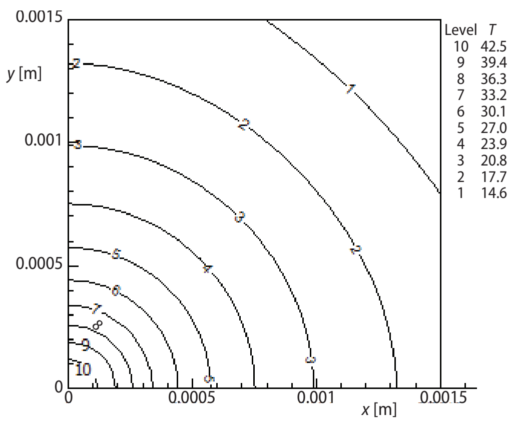

Figure 5. Temperature distribution at steady-state situation at *z* = 0, the slab is one end pumped by Gaussian beam, h_c = 5000 W/m^{2}K, h_a = 27.5 W/m^{2}K

steady-state temperature distribution could be reached. This phenomenon is explainable since the time of reaching steady-state depends mainly on the amount of heat that induced to the slab and the boundary conditions and since these parameters are constant then one can expect that the changing of beam profile has no effect on response time, fig. 7.

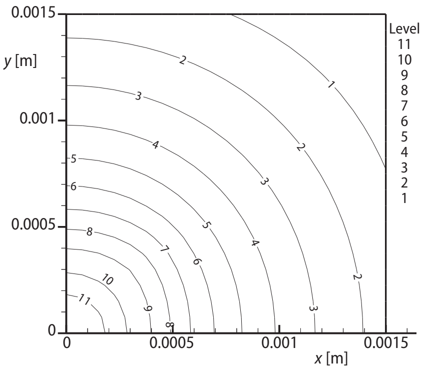

Figure 6. Temperature distribution at steady-state situation at $z = 0$ where the maximum temperature is 27.67 ºC, the slab is one end pumped by top-hat beam, $h_c = 20000$ W/m²K, and $h_a = 27.5$ W/m²K

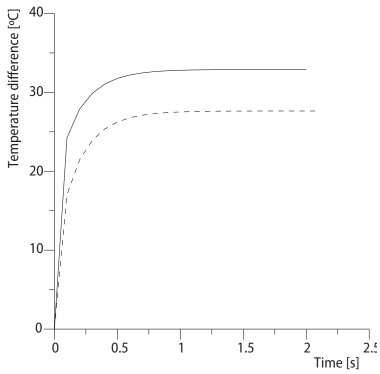

Figure 7. Temperature history of maximum temperature is laser slab at $x, y, z = 0$ where $h_c = 20000$ W/m²K, solid line for Gaussian beam distribution, dashed line for top-hat beam where its radius is equal to $2w_p$

The pumping power can be divided between the two ends of the slab, where the same output laser power could be obtained comparing with one end pumped slab. No effect on the time required to reach thermal equilibrium is observed as the pumping is changed from one to dual end pumping, this is due to the fact that the same energy is induced to the slab cooled by convection which has no effect on response time, fig. 8. Dividing the pumping power between the two ends of the slab for Gaussian or top hat pumping beam profile result in significant reduction in the temperature distribution through the slab for the aforementioned pumping profiles, this include of course the maximum temperature that could be reached in the slab (at the center of the end faces), see figs. 5 and 9.

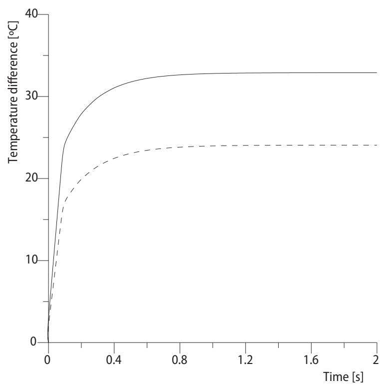

Figure 8. Temperature history of maximum temperature is laser slab where $h_c = 20000$ W/m²K, solid line for one end Gaussian beam pumping, dashed line for dual Gaussian end pumping

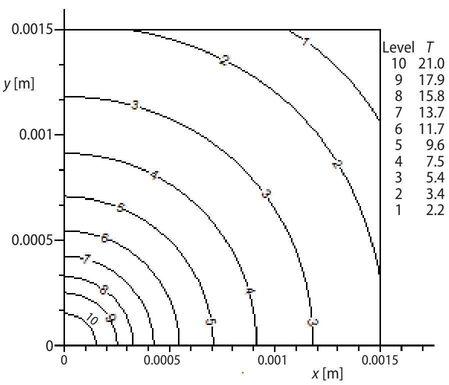

Figure 9. Temperature distribution at steady-state situation at $z = 0$, the slab is dual end pumped by Gaussian beam, $h_c = 20000$ W/m²K., and maximum temperature is 24 ºC

Conclusions

In this work, the integral transform method has been used to derive an analytical solution of transient temperature distribution in laser slab. The effect of different boundary conditions, pumping beam profile, pumping method on temperature distribution are obtained and the speed of response time for these conditions have been predicted. It is found that for the same total amount of pumping power, even that the dividing of the pumping power over the two laser slab ends reduces the temperature distribution in laser slab and it has no effect on the response time.

It is also found that the higher the lateral convection heat transfer coefficient, the lesser the time required to achieve thermal equilibrium. Also it is found that the profile of pumping beam profile has no effect on response time even though the top hat beam can reduce the temperature distribution through the laser slab.

The obtained results are compared with previously published data and good agreements are found which verified the used theory and method of solution. The obtained analytical solution can be very useful to designers in obtaining a device that could reach thermal equilibrium quickly where the designed laser parameters could be obtained and reduce the temperature distribution where good beam characteristic and safe operation conditions far away from failure stress could be obtained. It is worthwhile to mention that the anisotropic properties can be handled through some extension of the derived equation also slab rectangular cross section can be used instead of square cross section. These are aims of the new paper for the authors.

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