Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x).$$

Exercises

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \dfrac{4x^3}{3} - x + 2e^x$
6. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{4}$
7. $w = 3z^{-2} - \dfrac{1}{z}$
8. $s = -2t^{-1} + \dfrac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$
12. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$

In Exercises 13–16, find y' **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$
14. $y = (2x + 3)(5x^2 - 4x)$
15. $y = (x^2 + 1)\left(x + 5 + \dfrac{1}{x}\right)$
16. $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 17–40.

17. $y = \dfrac{2x + 5}{3x - 2}$
18. $z = \dfrac{4 - 3x}{3x^2 + x}$
19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$
20. $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$
21. $v = (1 - t)(1 + t^2)^{-1}$
22. $w = (2x - 7)^{-1}(x + 5)$
23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$
24. $u = \dfrac{5x + 1}{2\sqrt{x}}$
25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$
26. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$
27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$
28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$
29. $y = 2e^{-x} + e^{3x}$
30. $y = \dfrac{x^2 + 3e^x}{2e^x - x}$
31. $y = x^3e^x$
32. $w = re^{-r}$
33. $y = x^{9/4} + e^{-2x}$
34. $y = x^{-3/5} + \pi^{3/2}$
35. $s = 2t^{3/2} + 3e^2$
36. $w = \dfrac{1}{z^{1.4}} + \dfrac{\pi}{\sqrt{z}}$
37. $y = \sqrt[7]{x^2} - x^e$
38. $y = \sqrt[3]{x^{9.6}} + 2e^{1.3}$
39. $r = \dfrac{e^s}{s}$
40. $r = e^\theta\left(\dfrac{1}{\theta^2} + \theta^{-\pi/2}\right)$

Find the derivatives of all orders of the functions in Exercises 41–44

41. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$
42. $y = \dfrac{x^5}{120}$
43. $y = (x - 1)(x^2 + 3x - 5)$
44. $y = (4x^3 + 3x)(2 - x)$

Find the first and second derivatives of the functions in Exercis 45–52.

45. $y = \dfrac{x^3 + 7}{x}$
46. $s = \dfrac{t^2 + 5t - 1}{t^2}$
47. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$
48. $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$
49. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$
50. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$
51. $w = 3z^2e^{2z}$
52. $w = e^z(z - 1)(z^2 + 1)$

Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$

$$= 2x - \cos x$$

(b) $y = e^x \sin x$:

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) \sin x \qquad \text{Product Rule}$$

$$= e^x \cos x + e^x \sin x$$

$$= e^x (\cos x + \sin x)$$

(c) $y = \frac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x \cos x - \sin x}{x^2}$$

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The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

EXAMPLE 2 We find derivatives of the cosine function in combinations with other functions.

(a) $y = 5e^x + \cos x$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) && \text{Sum Rule} \\ &= 5e^x - \sin x \end{aligned}$$

(b) $y = \sin x \cos x$:

$$\begin{aligned} \frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule} \\ &= \sin x(-\sin x) + \cos x(\cos x) \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule} \\ &= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\ &= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{1 - \sin x} \end{aligned}$$

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The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

Find dy/dx

11. $y = \dfrac{\cot x}{1 + \cot x}$

12. $y = \dfrac{\cos x}{1 + \sin x}$

13. $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$

14. $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$

15. $y = x^2 \sin x + 2x \cos x - 2 \sin x$

16. $y = x^2 \cos x - 2x \sin x - 2 \cos x$

17. $f(x) = x^3 \sin x \cos x$

18. $g(x) = (2 - x) \tan^2 x$

In Exercises 19–22, find ds/dt.

19. $s = \tan t - e^{-t}$

20. $s = t^2 - \sec t + 5e^t$

21. $s = \dfrac{1 + \csc t}{1 - \csc t}$

22. $s = \dfrac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

23. $r = 4 - \theta^2 \sin \theta$

24. $r = \theta \sin \theta + \cos \theta$

25. $r = \sec \theta \csc \theta$

26. $r = (1 + \sec \theta) \sin \theta$

In Exercises 27–32, find dp/dq.

27. $p = 5 + \dfrac{1}{\cot q}$

28. $p = (1 + \csc q) \cos q$

29. $p = \dfrac{\sin q + \cos q}{\cos q}$

30. $p = \dfrac{\tan q}{1 + \tan q}$

31. $p = \dfrac{q \sin q}{q^2 - 1}$

32. $p = \dfrac{3q + \tan q}{q \sec q}$

33. Find y'' if

a. $y = \csc x$.

b. $y = \sec x$.

34. Find $y^{(4)} = d^4 y/dx^4$ if

a. $y = -2 \sin x$.

b. $y = 9 \cos x$.