

Differentiation rule

DIFFERENTIATION RULES

General Formulas

Assume u and v are differentiable functions of x .

Constant: $\frac{d}{dx}(c) = 0$

Sum: $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Difference: $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$

Constant Multiple: $\frac{d}{dx}(cu) = c \frac{du}{dx}$

Product: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Power: $\frac{d}{dx}x^n = nx^{n-1}$

Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{csc}^2 x \quad \frac{d}{dx}(\operatorname{csc} x) = -\operatorname{csc} x \cot x$$

Exponential and Logarithmic Functions

$$\frac{d}{dx}e^x = e^x \quad \frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx}a^x = a^x \ln a \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2} \quad \frac{d}{dx}(\operatorname{csc}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \quad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \quad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \quad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \quad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}}$$

Parametric Equations

If $x = f(t)$ and $y = g(t)$ are differentiable, then

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx .

EXAMPLE Find dy/dx if $y^2 = x^2 + \sin xy$ (Figure 3.31).

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

Differentiate both sides with respect to x ...

$$2y \frac{dy}{dx} = 2x + (\cos xy) \frac{d}{dx}(xy)$$

... treating y as a function of x and using the Chain Rule.

$$2y \frac{dy}{dx} = 2x + (\cos xy) \left(y + x \frac{dy}{dx} \right)$$

Treat xy as a product.

$$2y \frac{dy}{dx} - (\cos xy) \left(x \frac{dy}{dx} \right) = 2x + (\cos xy)y$$

Collect terms with dy/dx .

$$(2y - x \cos xy) \frac{dy}{dx} = 2x + y \cos xy$$

$$\frac{dy}{dx} = \frac{2x + y \cos xy}{2y - x \cos xy}$$

Solve for dy/dx .

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–16.

- $x^2y + xy^2 = 6$
- $x^3 + y^3 = 18xy$
- $2xy + y^2 = x + y$
- $x^3 - xy + y^3 = 1$
- $x^2(x - y)^2 = x^2 - y^2$
- $(3xy + 7)^2 = 6y$
- $y^2 = \frac{x-1}{x+1}$
- $x^3 = \frac{2x-y}{x+3y}$
- $x = \tan y$
- $xy = \cot(xy)$
- $x + \tan(xy) = 0$
- $x^4 + \sin y = x^3y^2$
- $y \sin\left(\frac{1}{y}\right) = 1 - xy$
- $x \cos(2x + 3y) = y \sin x$
- $e^{2x} = \sin(x + 3y)$
- $e^{xy} = 2x + 2y$

Find $dr/d\theta$ in Exercises 17–20.

- $\theta^{1/2} + r^{1/2} = 1$
- $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$
- $\sin(r\theta) = \frac{1}{2}$
- $\cos r + \cot \theta = e^{r\theta}$

Second Derivatives

In Exercises 21–26, use implicit differentiation to find dy/dx and then d^2y/dx^2 .

- $x^2 + y^2 = 1$
- $x^{2/3} + y^{2/3} = 1$
- $y^2 = e^{x^2} + 2x$
- $y^2 - 2x = 1 - 2y$
- $2\sqrt{y} = x - y$
- $xy + y^2 = 1$
- If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.
- If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

In Exercises 29 and 30, find the slope of the curve at the given points.

- $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
- $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

Slopes, Tangents, and Normals

In Exercises 31–40, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

- $x^2 + xy - y^2 = 1$, $(2, 3)$
- $x^2 + y^2 = 25$, $(3, -4)$