LIMITS AND CONTINUITY

Mathematicians of the seventeenth century were keenly interested in the study of motion for objects on or near the earth and the motion of planets and stars. This study involved both the speed of the object and its direction of motion at any instant, and they knew the direction was tangent to the path of motion. The concept of a limit is fundamental to finding the velocity of a moving object and the tangent to a curve. In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a .precise way to distinguish between these behaviors

Limits of Function Values

Frequently when studying a function y=f(x) , we find ourselves interested in the function's behavior *near* a particular point , but not *at* x_o

the limits law

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

Examples

Find the following limits

$$\textbf{(a)}\ \lim_{x\to c}(x^3+4x^2-3) \qquad \textbf{(b)}\ \lim_{x\to c}\frac{x^4+x^2-1}{x^2+5} \qquad \textbf{(c)}\ \lim_{x\to -2}\sqrt{4x^2-3}$$

$$\textbf{(a)}\ \lim_{x\to c}(x^3+4x^2-3) = \lim_{x\to c}x^3 + \lim_{x\to c}4x^2 - \lim_{x\to c}3 = c^3+4c^2-3$$

$$\textbf{(b)}\ \lim_{x\to c}\frac{x^4+x^2-1}{x^2+5} = \frac{\lim_{x\to c}(x^4+x^2-1)}{\lim_{x\to c}(x^2+5)} = \frac{\lim_{x\to c}x^4+\lim_{x\to c}x^2-\lim_{x\to c}1}{\lim_{x\to c}x^2+\lim_{x\to c}5} = \frac{c^4+c^2-1}{c^2+5}$$

$$\textbf{(c)}\ \lim_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim_{x\to -2}(4x^2-3)} = \sqrt{\lim_{x\to -2}4x^2-\lim_{x\to -2}3} = \sqrt{4(-2)^2-3} = \sqrt{16-3} = \sqrt{13}$$

Evaluate the following limit

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}.$$

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} = \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)} = \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} = \frac{1}{\sqrt{x^2+100}+10}.$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10} = \frac{1}{\sqrt{0^2+100}+10} = \frac{1}{20} = 0.05.$$

Den

$x =$

The sandwich theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (Figure 2.12). You will find a proof in Appendix 4.

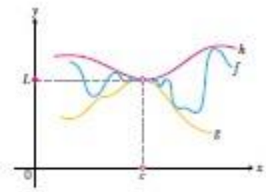


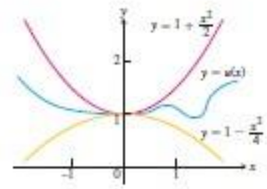
FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.



EXAMPLE Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \quad \text{for all } x \ne 0,$$

find $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x\to 0}(1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x\to 0}(1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$ (Figure 2.13).

all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

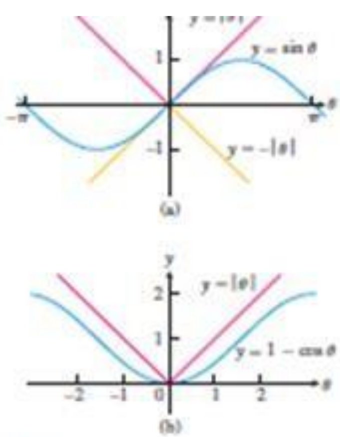


FIGURE The Sandwich Theorem confirms the limits in Example 11.

EXAMPLE The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta\to 0} \sin\theta = 0$ **(b)** $\lim_{\theta\to 0} \cos\theta = 1$

(c) For any function f, $\lim_{x\to c} |f(x)| = 0$ implies $\lim_{x\to c} f(x) = 0$.

Solution

(a) In Section 1.3 we established that $-|\theta| \le \sin\theta \le |\theta|$ for all θ (see Figure 2.14a). Since $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0}|\theta| = 0$, we have

$$\lim_{\theta\to 0} \sin\theta = 0.$$

(b) From Section 1.3, $0 \le 1 - \cos\theta \le |\theta|$ for all θ (see Figure 2.14b), and we have $\lim_{\theta\to 0}(1 - \cos\theta) = 0$ or

$$\lim_{\theta\to 0} \cos\theta = 1.$$

(c) Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x\to c} f(x) = 0$. ■