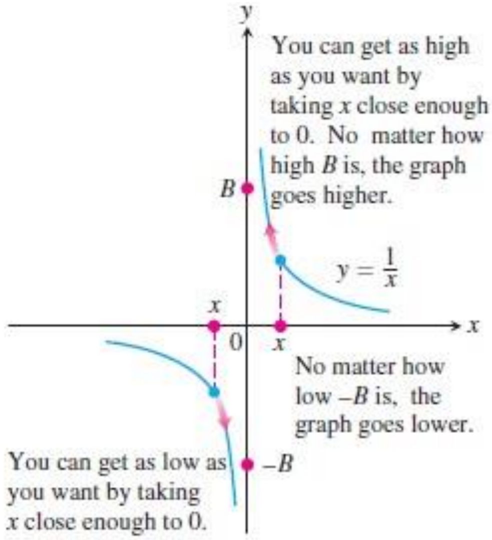
Infinite Limits

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still



DIFFERENTIATION RULES

General Formulas

Assume u and v are differentiable functions of x.

Constant: $\frac{d}{dx}(c) = 0$

Sum: $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Difference: $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$

Constant Multiple: $\frac{d}{dx}(cu) = c\frac{du}{dx}$

Product: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Power: $\frac{d}{dx}x^n = nx^{n-1}$

Chain Rule: $\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad \frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$ $\qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$

Exponential and Logarithmic Functions

$\frac{d}{dx}e^x = e^x$ $\qquad \frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}a^x = a^x \ln a$ $\qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ $\qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$ $\qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2}$ $\qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$

Hyperbolic Functions

$\frac{d}{dx}(\sinh x) = \cosh x$ $\qquad \frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ $\qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$ $\qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$ $\qquad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}$ $\qquad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - x^2}}$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2}$ $\qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1 + x^2}}$

Parametric Equations

If $x = f(t)$ and $y = g(t)$ are differentiable, then

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

EXAMPLE 11 Find $\lim_{x\to 1^+} \frac{1}{x-1}$ and $\lim_{x\to 1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x-1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.60). Therefore, $y = 1/(x-1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \qquad \text{and} \qquad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $1/(x-1) \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $1/(x-1) \to -\infty$. ■

EXAMPLE 12 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \qquad \text{as} \qquad x \to 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.61). This means that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty. \qquad ■$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0} (1/x^2) = \infty$.

Examples

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

(b) $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$

(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$

(e) $\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist.

(f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

Limits and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0$, and 1. Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \leq -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

Finding Limits

In Exercises 9–28, find the limit or explain why it does not exist.

9. $\lim \dfrac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

a. as $x \to 0$ **b.** as $x \to 2$

10. $\lim \dfrac{x^2 + x}{x^5 + 2x^4 + x^3}$

a. as $x \to 0$ **b.** as $x \to -1$

11. $\lim\limits_{x \to 1} \dfrac{1 - \sqrt{x}}{1 - x}$

12. $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x^4 - a^4}$

13. $\lim\limits_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

14. $\lim\limits_{x \to 0} \dfrac{(x + h)^2 - x^2}{h}$

15. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

16. $\lim\limits_{x \to 0} \dfrac{(2 + x)^3 - 8}{x}$

17. $\lim\limits_{x \to 1} \dfrac{x^{1/3} - 1}{\sqrt{x} - 1}$

18. $\lim\limits_{x \to 64} \dfrac{x^{2/3} - 16}{\sqrt{x} - 8}$

19. $\lim\limits_{x \to 0} \dfrac{\tan(2x)}{\tan(\pi x)}$

20. $\lim\limits_{x \to \pi^-} \csc x$

21. $\lim\limits_{x \to \pi} \sin\left(\dfrac{x}{2} + \sin x\right)$

22. $\lim\limits_{x \to \pi} \cos^2(x - \tan x)$

23. $\lim\limits_{x \to 0} \dfrac{8x}{3 \sin x - x}$

24. $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{\sin x}$

Generalized Limits Involving $\dfrac{\sin \theta}{\theta}$

The formula $\lim_{\theta \to 0}(\sin \theta)/\theta = 1$ can be generalized. If $\lim_{x \to c} f(x) = 0$ and $f(x)$ is never zero in an open interval containing the point $x = c$, except possibly c itself, then

$$\lim_{x \to c} \frac{\sin f(x)}{f(x)} = 1.$$

Here are several examples.

a. $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x^2} = 1$

b. $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x} = \lim\limits_{x \to 0} \dfrac{\sin x^2}{x^2} \lim\limits_{x \to 0} \dfrac{x^2}{x} = 1 \cdot 0 = 0$

c. $$\lim_{x\to-1}\frac{\sin(x^2-x-2)}{x+1}=\lim_{x\to-1}\frac{\sin(x^2-x-2)}{(x^2-x-2)}\cdot$$
$$\lim_{x\to-1}\frac{(x^2-x-2)}{x+1}=1\cdot\lim_{x\to-1}\frac{(x+1)(x-2)}{x+1}=-3$$

d. $$\lim_{x\to1}\frac{\sin\left(1-\sqrt{x}\right)}{x-1}=\lim_{x\to1}\frac{\sin\left(1-\sqrt{x}\right)}{1-\sqrt{x}}\frac{1-\sqrt{x}}{x-1}=$$
$$1\cdot\lim_{x\to1}\frac{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}{(x-1)\left(1+\sqrt{x}\right)}=\lim_{x\to1}\frac{1-x}{(x-1)\left(1+\sqrt{x}\right)}=-\frac{1}{2}$$

Find the limits in Exercises 25–30.

25. $\displaystyle\lim_{x\to0}\frac{\sin(1-\cos x)}{x}$

26. $\displaystyle\lim_{x\to0^+}\frac{\sin x}{\sin\sqrt{x}}$

27. $\displaystyle\lim_{x\to0}\frac{\sin(\sin x)}{x}$

28. $\displaystyle\lim_{x\to0}\frac{\sin(x^2+x)}{x}$

29. $\displaystyle\lim_{x\to2}\frac{\sin(x^2-4)}{x-2}$

30. $\displaystyle\lim_{x\to9}\frac{\sin\left(\sqrt{x}-3\right)}{x-9}$