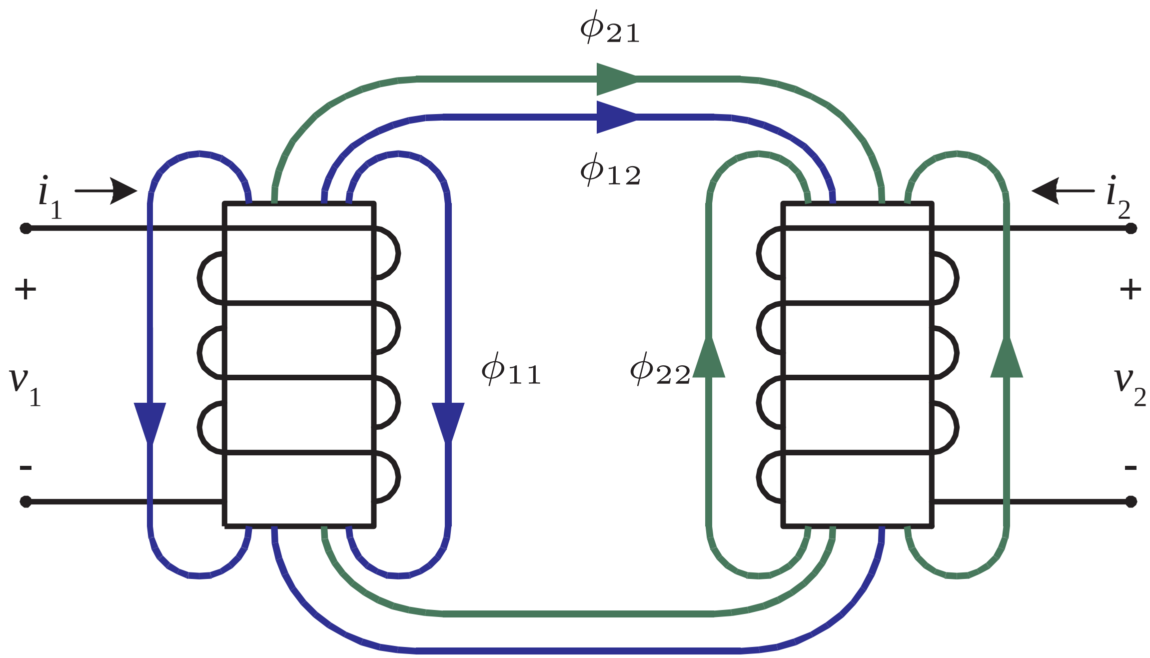
Magnetically Coupled Circuits Overview

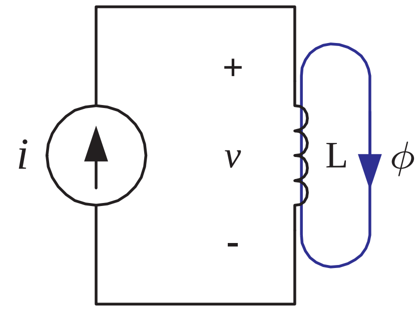
- Mutual Inductance
- Energy in Coupled Coils
- Linear Transformers
- Ideal Transformers

Introduction to Magnetically Coupled Circuits



- Magnetically coupled coils are conceptually similar to two inductors that have a shared (coupled) magnetic field
- Not all of the magnetic field is shared
- Magnetic coupling is widely used in power systems

Magnetically Coupled Coil



Faraday's Law:

$$v = N\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

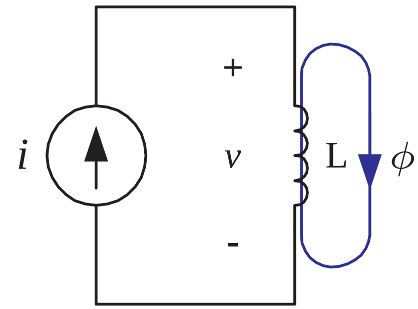
$$\phi = N\mathcal{P}i$$

$$v = N\frac{\mathrm{d}\left(N\mathcal{P}i\right)}{\mathrm{d}t} = N^2\mathcal{P}\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

where

$v =$ voltage in volts (V)
$N =$ number of turns
$\phi =$ magnetic flux in webers (Wb)
$t =$ time in seconds (s)
$\mathcal{P} =$ permeance of the flux space
$i =$ current in amperes (A)
$L =$ inductance in henrys (H)

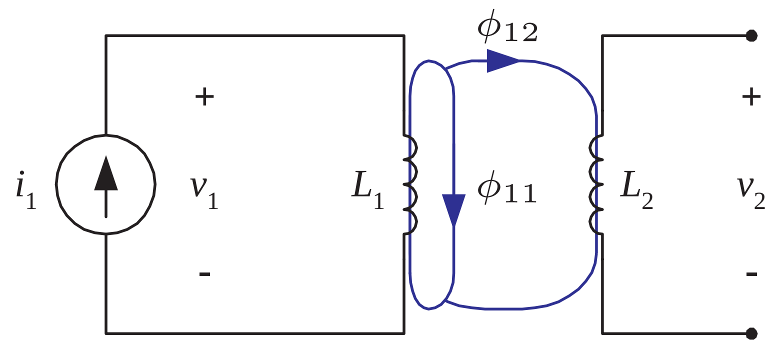
Magnetically Coupled Coil



$$v = N\frac{\mathrm{d}\phi}{\mathrm{d}t} = (N^2\mathcal{P})\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

- The flux (& current) have to change to induce a voltage
- The relationship between the flux and the current is constant
- Consistent with what we already know about inductors
- L is proportional to N^2

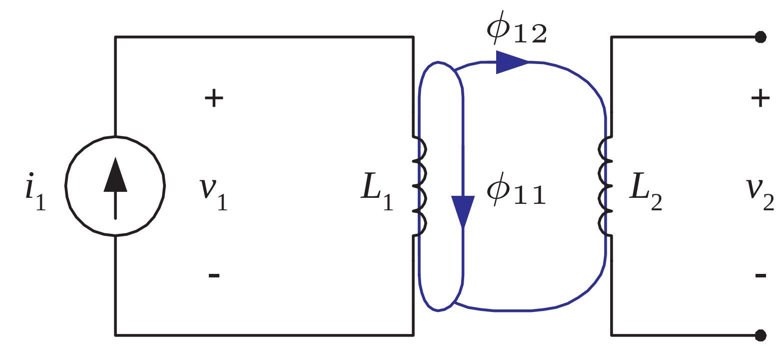
Mutual Inductance



$$\phi_1 = \phi_{11} + \phi_{12}$$

- We can decompose the magnetic flux induced in one coil into two components
- ϕ_1 is the total flux produced in coil 1
- ϕ_{11} is the portion of this flux that links only coil 1
- ϕ_{12} links both coil 1 and coil 2
- The coils are not connected electrically

Mutual Inductance Continued

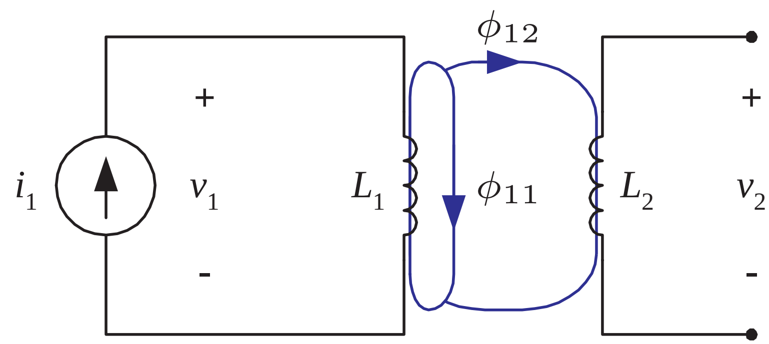


$$v_1 = N_1 \frac{d\phi_1}{dt} = N_1 \frac{d(N_1 \mathcal{P}_1 i_1)}{dt} = L_1 \frac{di_1}{dt}$$

$$v_2 = \pm N_2 \frac{d\phi_{12}}{dt} = \pm N_2 \frac{d(N_1 \mathcal{P}_{12} i_1)}{dt} = \pm (N_2 N_1 \mathcal{P}_{12}) \frac{di_1}{dt}$$

$$= \pm M_{21} \frac{di_1}{dt}$$

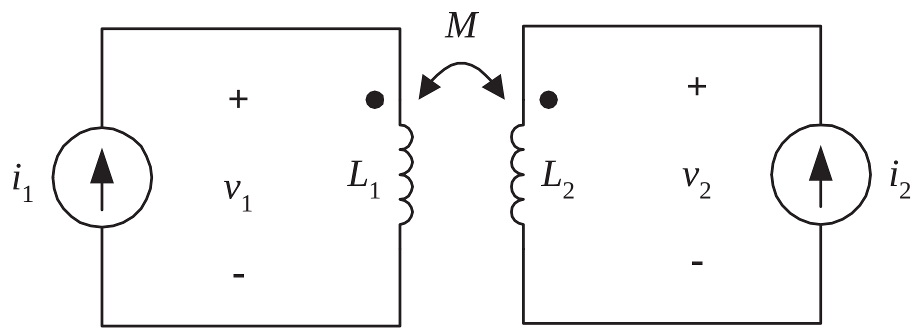
Mutual Inductance Continued 2



$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} \qquad v_2 = \pm M_{21} \frac{\mathrm{d}i_1}{\mathrm{d}t} \qquad M_{21} = N_2 N_1 \mathcal{P}_{12}$$

- M_{21}: the mutual inductance of coil 2 with respect to coil 1
- Note that v_2 is the open circuit voltage
- What if a current was applied to coil 2 as well?
- Superposition applies

Mutual Inductance: Two Sources

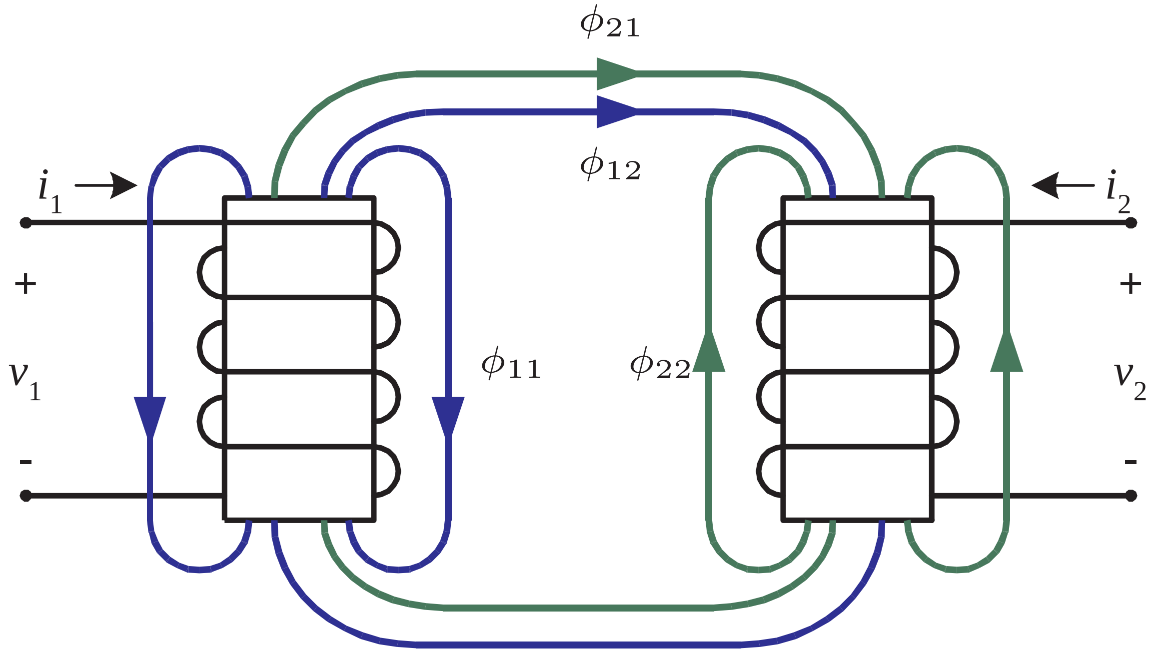


$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} \pm M_{12} \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

$$v_2 = \pm M_{21} \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

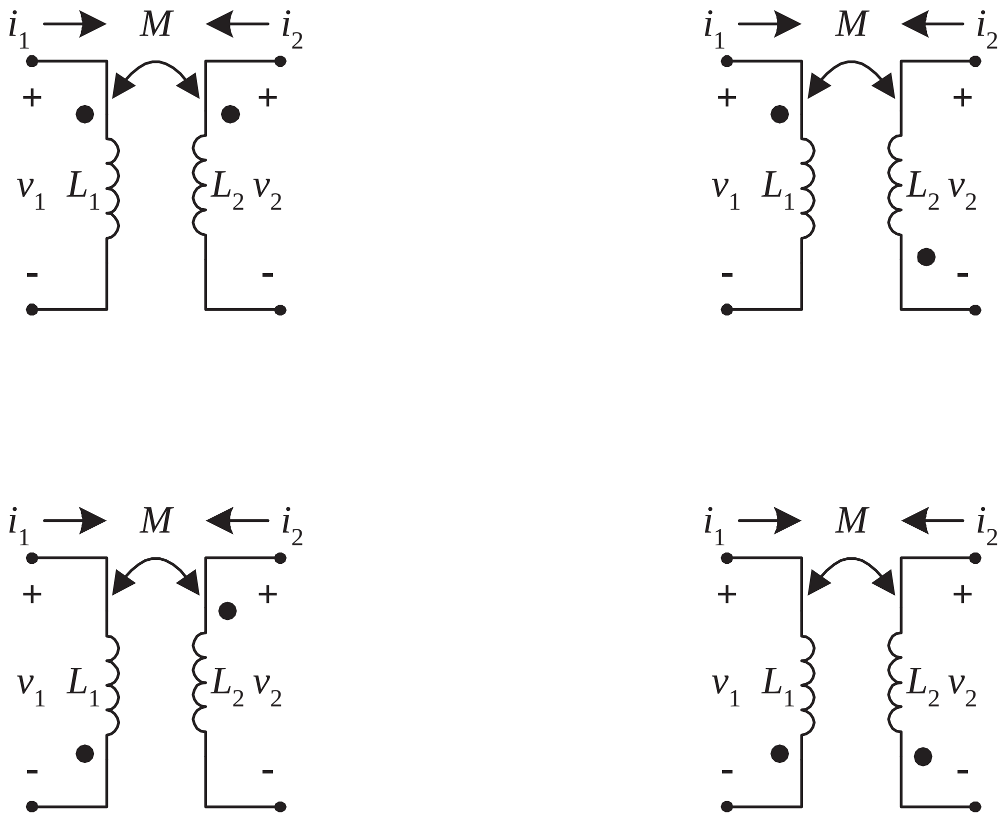
- We will assume $M_{21} = M_{12} = M$
- If assumption holds, the coils are called a **linear transformer**
- M is called the **mutual inductance**
- Like inductors, is measured in units of henrys (H)
- Polarity of coupling term depends on how the coils are wound

Linear Transformer: The Dot Convention



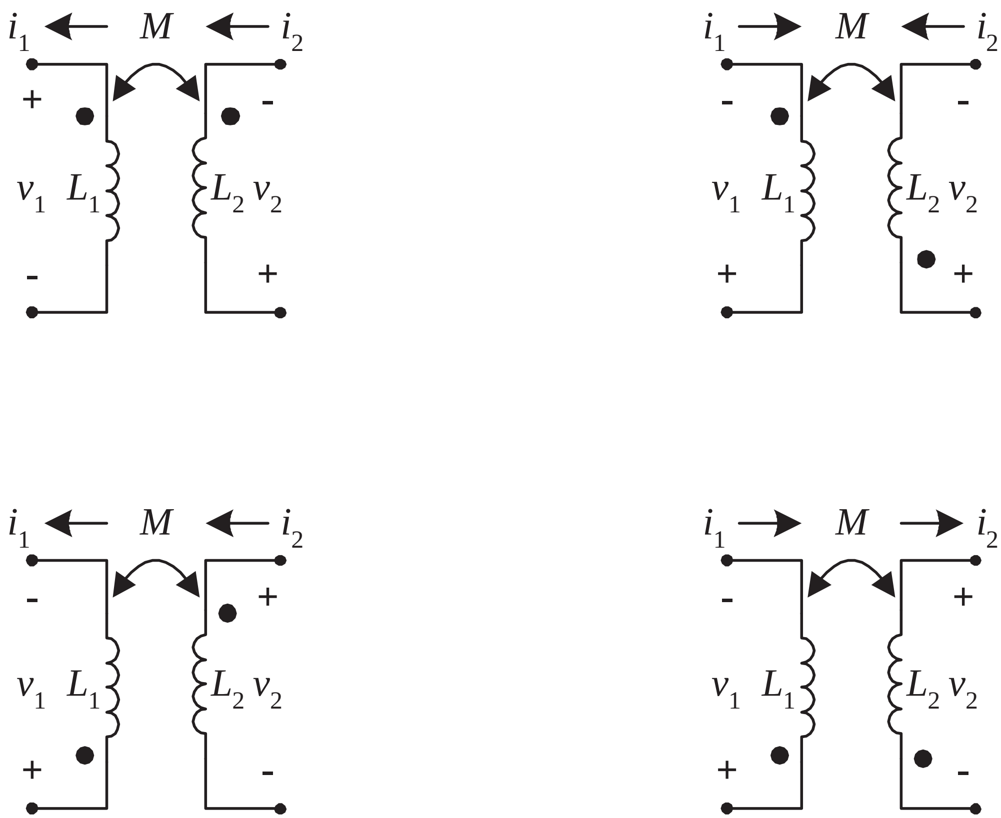
- The dot convention determines the polarity of the coupling
- **Dot Convention**: If a current *enters* a dotted terminal, it induces a *positive* voltage at the dotted terminal of the second coil
- If a current *leaves* a dotted terminal, it induces a *negative* voltage at the dotted terminal of the second coil

Example 1: The Dot Convention



Write the defining equations for each of the circuits shown above.

Example 2: The Dot Convention



Write the defining equations for each of the circuits shown above.

Mutual Inductance & Self Inductance

$$L_1 = N_1^2 \mathcal{P}_1 \qquad \mathcal{P}_1 = \mathcal{P}_{11} + \mathcal{P}_{12}$$

$$L_2 = N_2^2 \mathcal{P}_2 \qquad \mathcal{P}_2 = \mathcal{P}_{22} + \mathcal{P}_{21}$$

$$\begin{aligned} L_1 L_2 &= N_1^2 N_2^2 \mathcal{P}_1 \mathcal{P}_2 \\ L_1 L_2 &= N_1^2 N_2^2 (\mathcal{P}_{11} + \mathcal{P}_{12})(\mathcal{P}_{22} + \mathcal{P}_{21}) \end{aligned}$$

Since $M_{12} = M_{21}$ for a linear transformer, $\mathcal{P}_{12} = \mathcal{P}_{21}$ and

$$\begin{aligned} L_1 L_2 &= N_1^2 N_2^2 \mathcal{P}_{12}^2 \left(1 + \frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right)\left(1 + \frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right) \\ &= M^2 \frac{1}{k^2} \\ M &= k\sqrt{L_1 L_2} \end{aligned}$$

Coefficient of Coupling (k)

$$\frac{1}{k^2} = \left(1 + \frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right)\left(1 + \frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right)$$

$$M = k\sqrt{L_1 L_2}$$

- k is called the **coefficient of coupling**
- Since $\frac{1}{k^2} \geq 1$, $k \leq 1$
- k is non-negative
- If two coils have no common flux, $k = 0$
- If both coils share *all* flux, $k = 1$
- It is physically impossible for $k = 1$, but some magnetic cores have k very close to 1

Linear Transformers: Energy

Suppose no energy is stored in the coils at $t = 0$ and over some period of time t_1 the current in coil 1 increases from 0 to I_1 while the current in coil 2 is zero, $i_2 = 0$. The energy stored in the coils over this period is given by

$$w_1 = \int_0^{t_1} v_1 i_1 \, \mathrm{d}\tau = \int_0^{t_1} \left(L \frac{\mathrm{d}i_1}{\mathrm{d}\tau} \right) i_1 \, \mathrm{d}\tau = L_1 \int_0^{I_1} i_1 \, \mathrm{d}i_1 = \tfrac{1}{2} L_1 I_1^2$$

Now suppose the current in coil 1 is held constant, $i_1 = I_1$, while the current in coil 2 increases from 0 to I_2.

$$\begin{aligned}
v_1 &= L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} \pm M \frac{\mathrm{d}i_2}{\mathrm{d}t} && = \pm M \frac{\mathrm{d}i_2}{\mathrm{d}t} \\
v_2 &= \pm M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} && = L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \\
p_2 &= v_1 I_1 + v_2 i_2 && = \pm M \frac{\mathrm{d}i_2}{\mathrm{d}t} I_1 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} i_2
\end{aligned}$$

Linear Transformers: Energy Continued

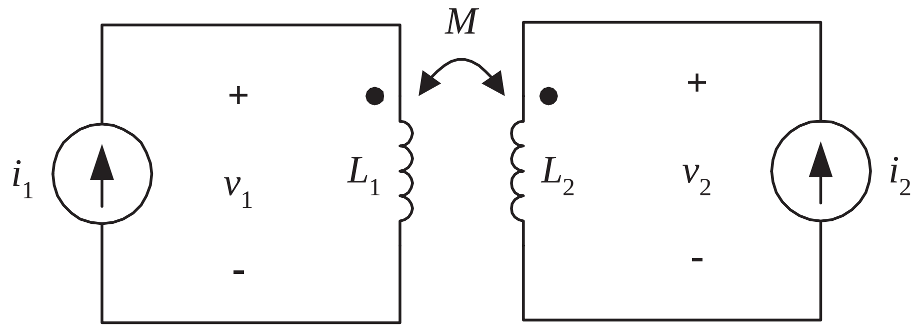
The energy stored in the coils during the second period is given by

$$\begin{aligned} w_2 &= \int_{t_1}^{t_2} p_2 \,\mathrm{d}\tau \\ &= \int_{t_1}^{t_2} \left(\pm M \frac{\mathrm{d}i_2}{\mathrm{d}\tau} I_1 + L_2 i_2 \frac{\mathrm{d}i_2}{\mathrm{d}\tau} \right) \mathrm{d}\tau \\ &= \pm M I_1 \int_0^{I_2} \mathrm{d}i_2 + L_2 \int_0^{I_2} i_2 \,\mathrm{d}i_2 \\ &= \pm M I_1 I_2 + \tfrac{1}{2} L_2 I_2^2 \end{aligned}$$

Then the total energy stored in magnetically coupled coils after the currents have been applied is given by

$$\begin{aligned} w &= w_1 + w_2 \\ &= \tfrac{1}{2} L_1 I_1^2 + \tfrac{1}{2} L_2 I_2^2 \pm M I_1 I_2 \end{aligned}$$

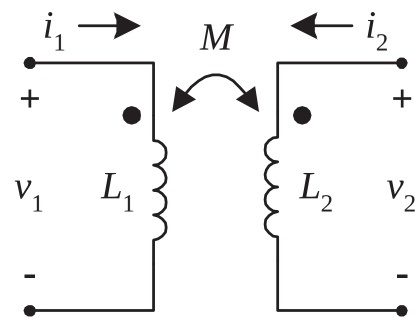
Linear Transformers: Energy Comments



$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$$

- The polarity of the shared term depends on how the coils are wound
- Can the energy stored ever be negative?
- Recall that $M = k\sqrt{L_1 L_2}$
- This limits the expression above to non-negative values only

Time-Domain Analysis



Time Domain

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} \pm M \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

$$v_2 = \pm M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

What is v_1 if $i_1 = A_1 \cos(\omega t)$ and $i_2 = 0$?

Sinusoidal Steady-State Analysis

What is v_1 if $i_1 = A_1 \cos(\omega t)$ and $i_2 = 0$?

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} \pm M \frac{\mathrm{d}i_2}{\mathrm{d}t} \qquad v_2 = \pm M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} \qquad v_2 = \pm M \frac{\mathrm{d}i_1}{\mathrm{d}t}$$

$$v_1 = \omega L_1 A_1 \left(-\sin(\omega t)\right) \qquad v_2 = \pm \omega M A_1 \left(-\sin(\omega t)\right)$$

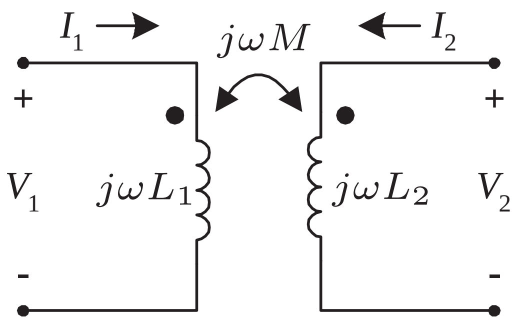
$$v_1 = \omega L_1 A_1 \cos(\omega t + 90^\circ) \qquad v_2 = \pm \omega M A_1 \cos(\omega t + 90^\circ)$$

What is the relationship in the phasor domain?

$$V_1 = j\omega L_1 I_1 \qquad V_2 = \pm j\omega M I_1$$

Superposition applies so if $i_2 = A_2 \cos(\omega t)$,

Sinusoidal Steady-State Analysis Continued

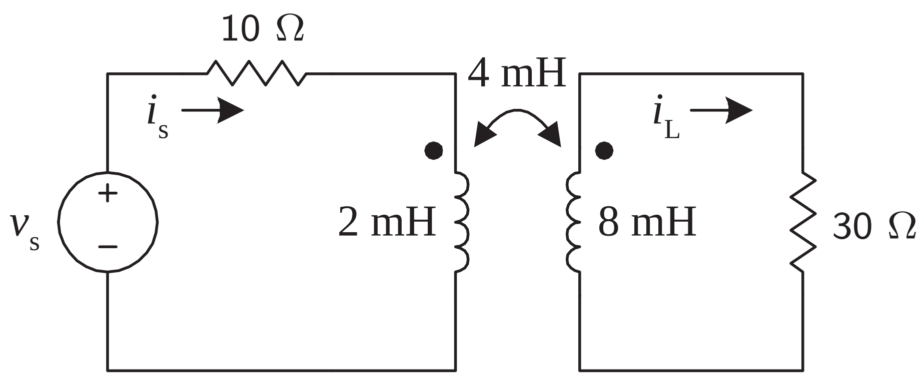


Frequency Domain (Phasors)

$$\begin{aligned} V_1 &= j\omega L_1 I_1 + j\omega M I_2 \\ V_2 &= j\omega M I_1 + j\omega L_2 I_2 \end{aligned}$$

The dot convention still applies (not shown)

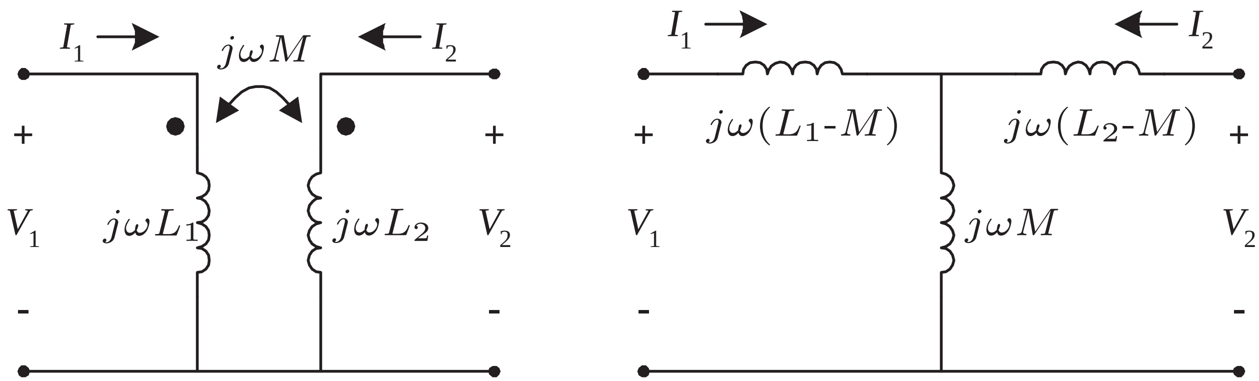
Example 3: Linear Transformers & Phasor Analysis



Find the steady-state expressions for the currents i_s and i_L when $v_s = 70\cos(5000t)$ V.

Example 3: Workspace

Phasor Analysis: T-Equivalent

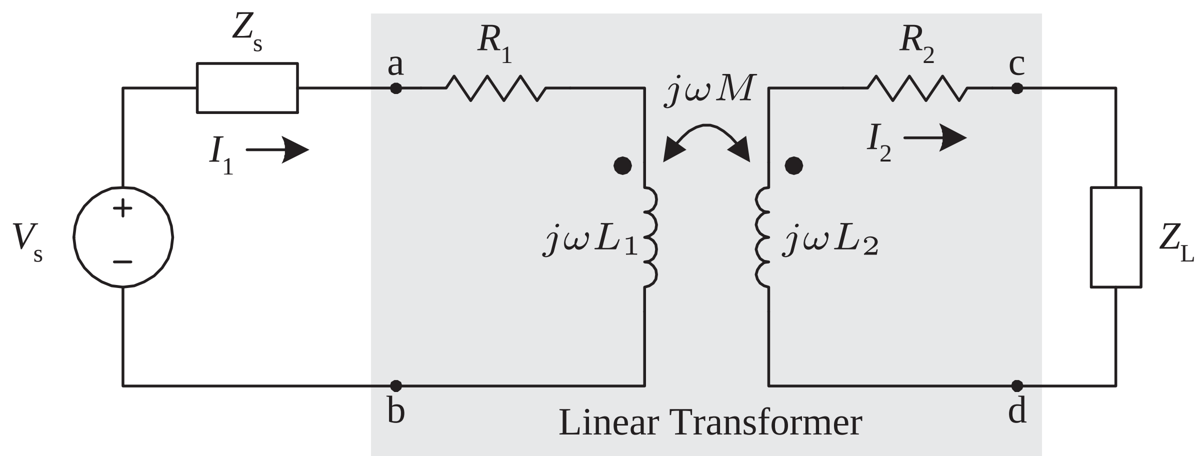


Frequency Domain (Phasors)

$$
\begin{aligned}
V_1 &= j\omega L_1 I_1 + j\omega M I_2 \\
V_2 &= j\omega M I_2 + j\omega L_2 I_2
\end{aligned}
$$

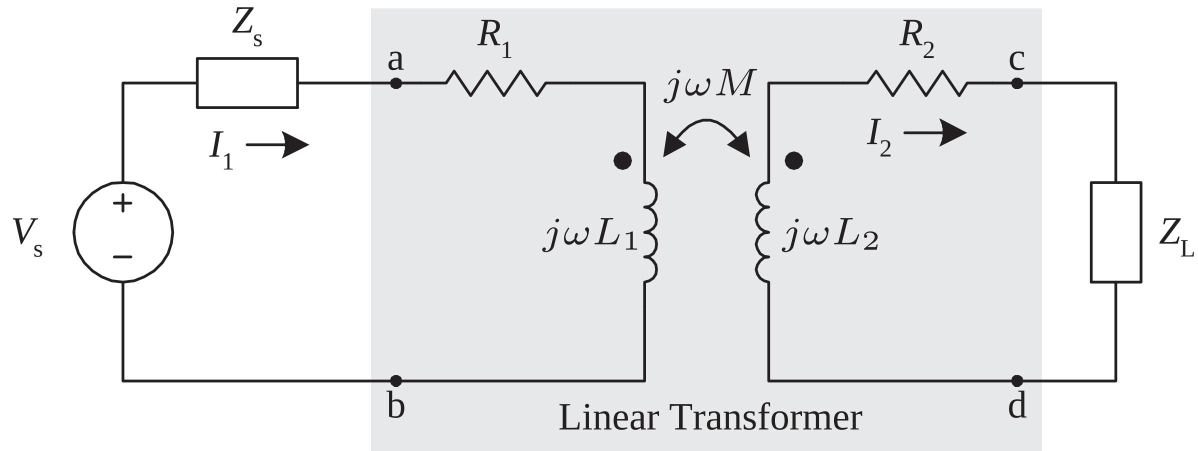
- The T-equivalent is only valid if bottom terminals are connected
- There is also a $\prod$-equivalent (see text)

Linear Transformer Analysis: Typical Configuration



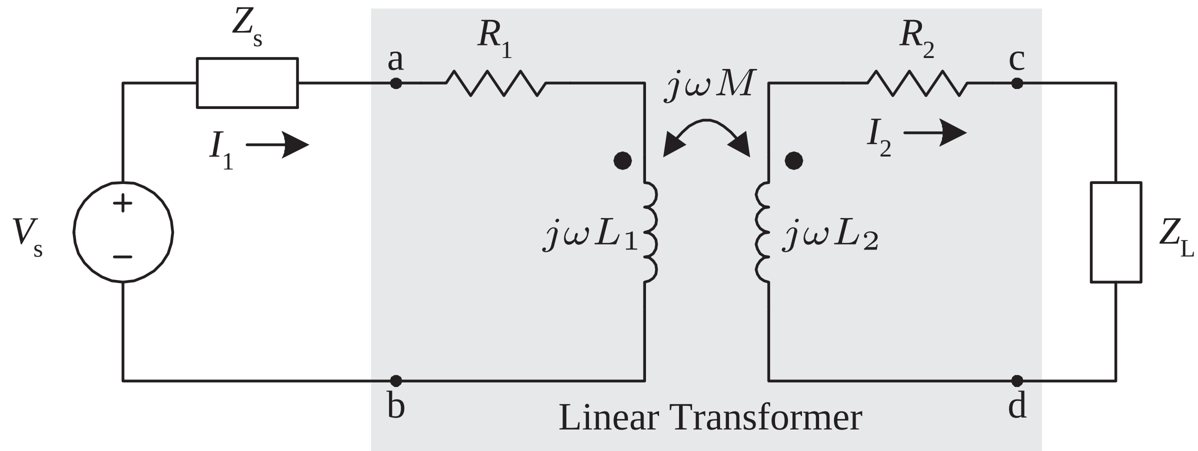
- Transformers are typically used in only few types of circuits
- The most common configuration connects a source to a load
- Useful to decrease (or increase) the voltage across the load
- Why not just use a voltage divider?
- Should know how to analyze this type of circuit thoroughly

Linear Transformer: Source–Load Analysis



R_1	=	Resistance of primary winding
R_2	=	Resistance of secondary winding
L_1	=	Self-inductance of primary
L_2	=	Self-inductance of secondary
M	=	Mutual inductance
Z_s	=	Source impedance
Z_L	=	Load impedance

Linear Transformer: Source–Load Analysis Continued

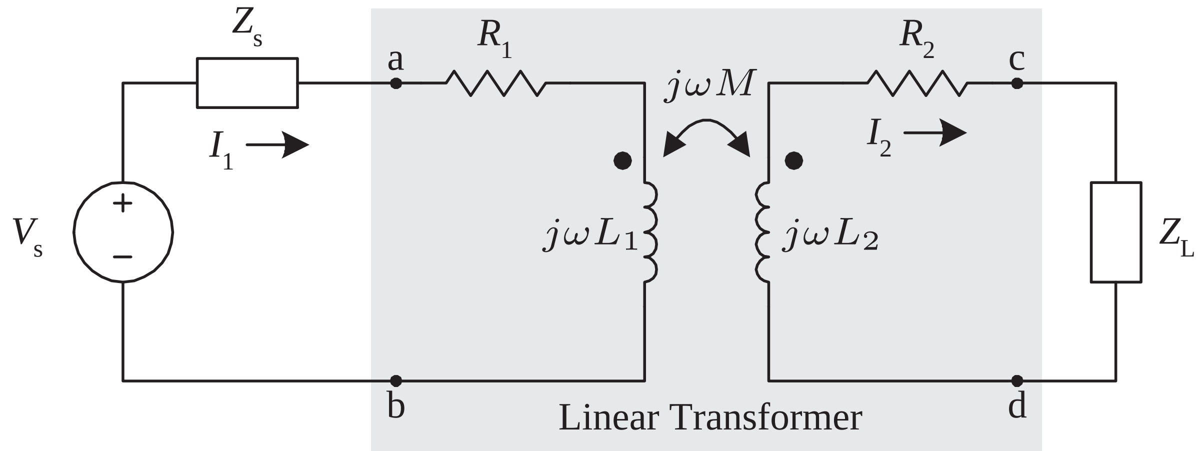


$$
\begin{aligned}
V_s &= (Z_s + R_1 + j\omega L_1)I_1 - j\omega M I_2 \\
0 &= -j\omega M I_1 + (R_2 + j\omega L_2 + Z_L)I_2 \\
Z_{11} &\triangleq Z_s + R_1 + j\omega L_1 \\
Z_{22} &\triangleq R_2 + j\omega L_2 + Z_L \\
V_s &= Z_{11} I_1 - j\omega M I_2 \\
0 &= -j\omega M I_1 + Z_{22} I_2
\end{aligned}
$$

Linear Transformer: Source–Load Analysis Continued 2

$$
\begin{aligned}
V_s &= Z_{11} I_1 - j\omega M I_2 \\
0 &= -j\omega M I_1 + Z_{22} I_2 \\
I_2 &= \frac{j\omega M}{Z_{22}} I_1 \\
V_s &= \left(Z_{11} + \frac{\omega^2 M^2}{Z_{22}} \right) I_1 \\
I_1 &= \frac{Z_{22}}{Z_{11} Z_{22} + \omega^2 M^2} V_s \\
I_2 &= \frac{j\omega M}{Z_{11} Z_{22} + \omega^2 M^2} V_s
\end{aligned}
$$

Linear Transformer: Source–Load Internal Impedance

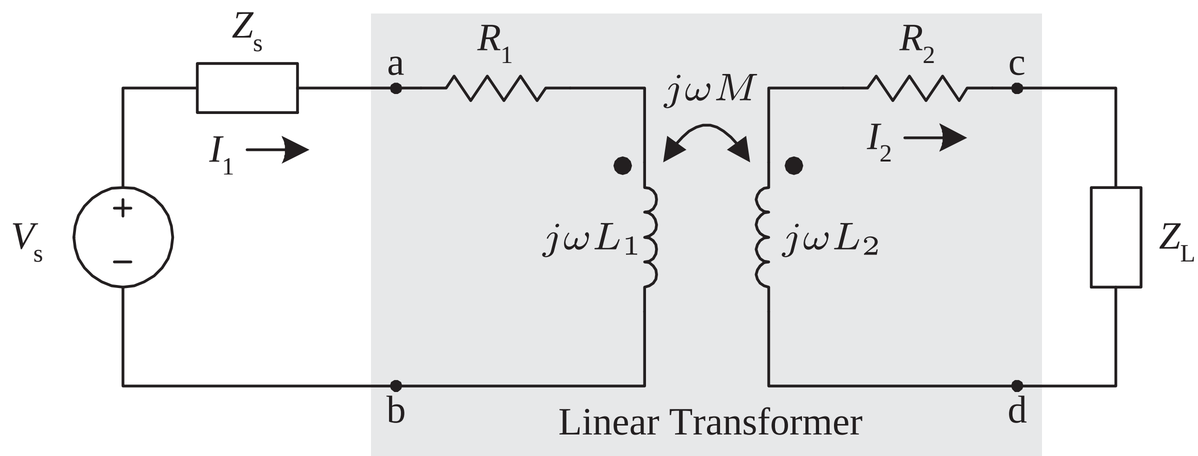


$$Z_i \triangleq \frac{V_s}{I_1} = \frac{Z_{11}Z_{22} + \omega^2 M^2}{Z_{22}}$$

$$= Z_{11} + \frac{\omega^2 M^2}{Z_{22}}$$

$$Z_{ab} = Z_i - Z_s = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{Z_{22}}$$

Linear Transformer: Source–Load Reflected Impedance

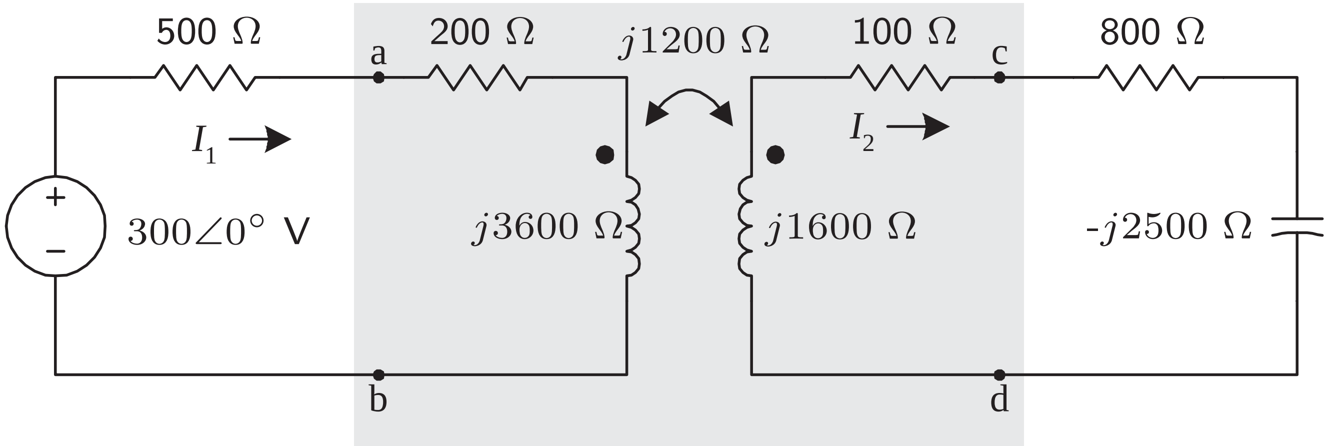


$$Z_{ab} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{Z_{22}}$$

$$Z_R = Z_{ab} - (R_1 + j\omega L_1)$$

$$= \frac{\omega^2 M^2}{Z_{22}} = \frac{\omega^2 M^2}{|Z_{22}|^2} Z_{22}^*$$

Example 4: Linear Transformers

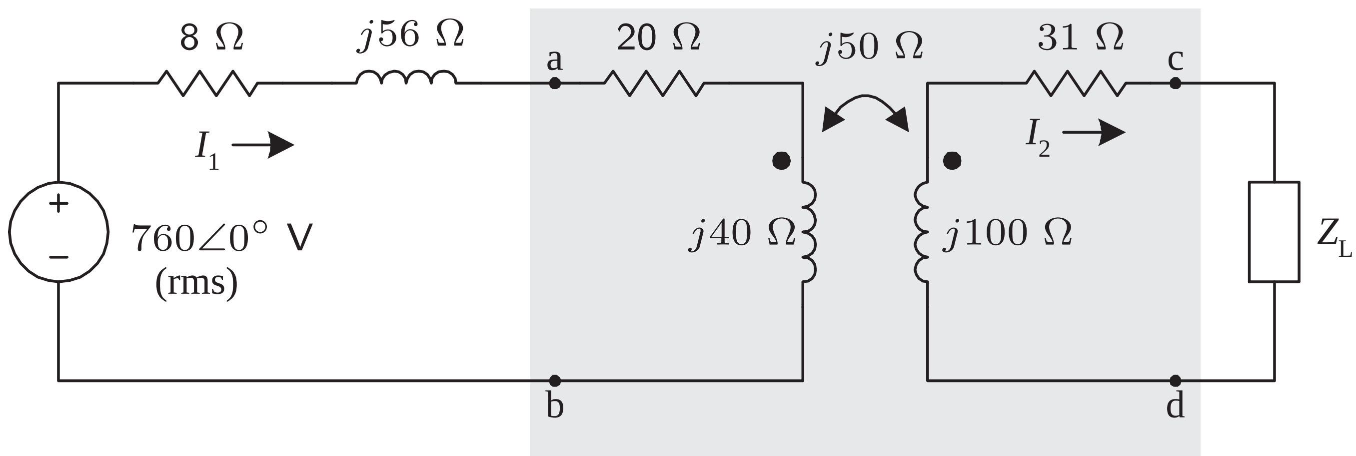


Find the following:

1. Self-impedance of primary & secondary circuits
2. Impedance reflected into the primary winding
3. Impedance seen looking into the primary terminals of the transformer
4. Thévenin equivalent with respect to the terminals c,d

Example 4: Workspace

Example 5: Linear Transformers



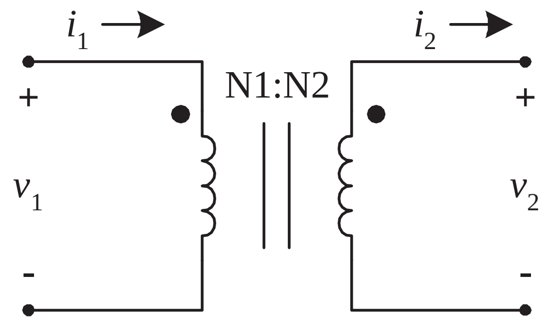
Find the following:

1. Thévenin equivalent with respect to the terminals c,d
2. If Z_L is set equal to Z_{eq}^*, what is I_1?
3. What is I_2?

Example 5: Workspace (1)

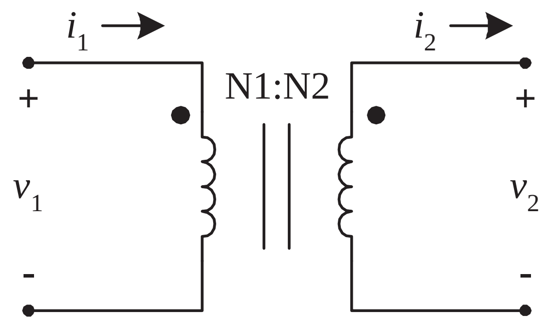
Example 5: Workspace (2)

Introduction to Ideal Transformers



- Ideal/Linear transformers are similar to ideal/real models of operational amplifiers
- Both ideal models make assumptions that simplify analysis
- Ideal approximation: all of the flux links both coils
- Ideal Assumptions
  - Large reactance: $L_1, L_2, M \rightarrow \infty$
  - Perfect coupling: $k \rightarrow 1$
  - Primary and secondary are lossless: $R_1 = R_2 = 0$

Ideal Transformer Analysis

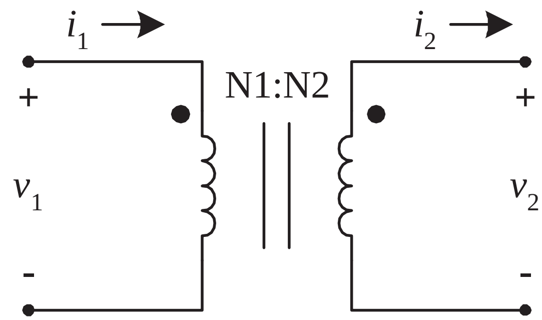


$$v_1 = N_1 \frac{\mathrm{d}\phi}{\mathrm{d}t} \qquad p_1 = p_2$$

$$v_2 = N_2 \frac{\mathrm{d}\phi}{\mathrm{d}t} \qquad v_1 i_1 = v_2 i_2$$

$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n \qquad \frac{i_1}{i_2} = \frac{v_2}{v_1} = n$$

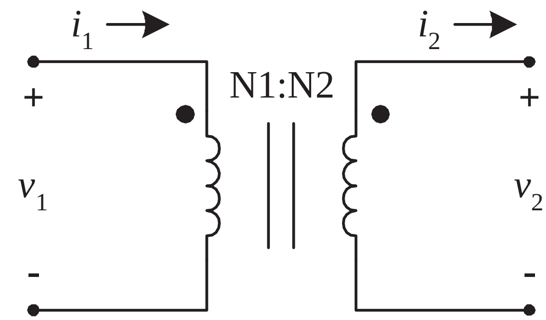
Ideal Transformers: Comments



$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n \qquad \frac{i_2}{i_1} = \frac{N_1}{N_2} = \frac{1}{n}$$

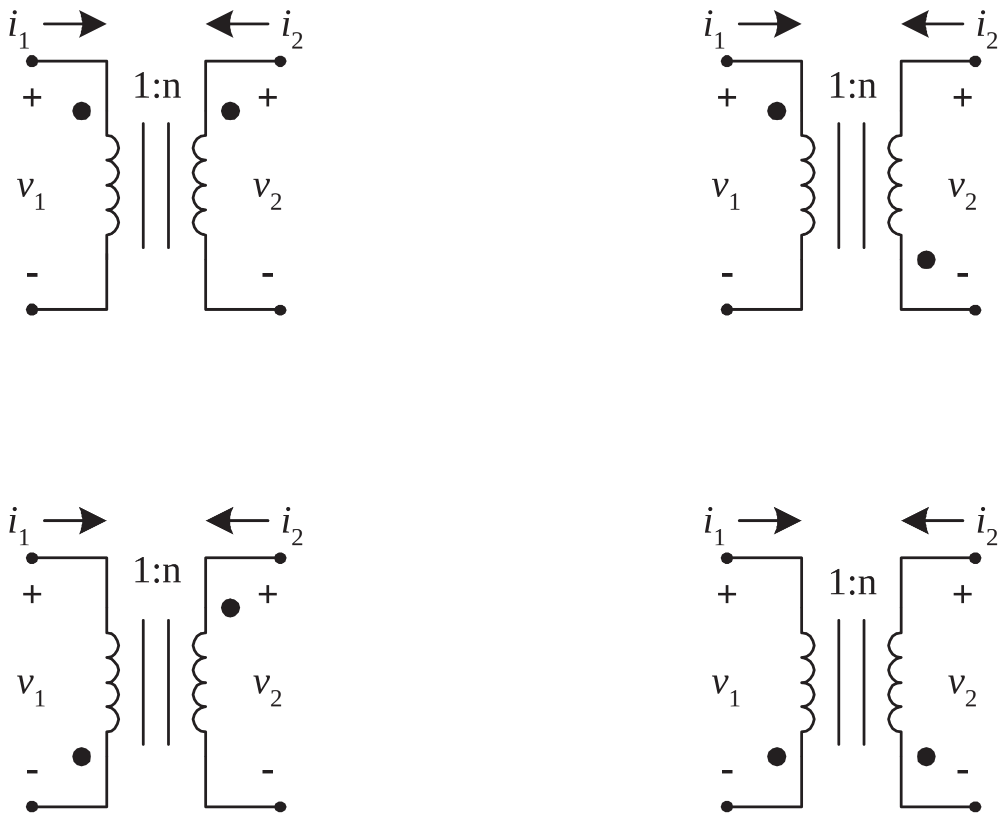
- Defining equations for ideal transformers do not include time
- The phasor domain equations are identical to the time domain
- The ideal transformer cannot store energy
- Note direction of secondary current
- Sometimes only the turns ratio is given: $N_2/N_1 = n$

Ideal Transformers: The Dot Convention



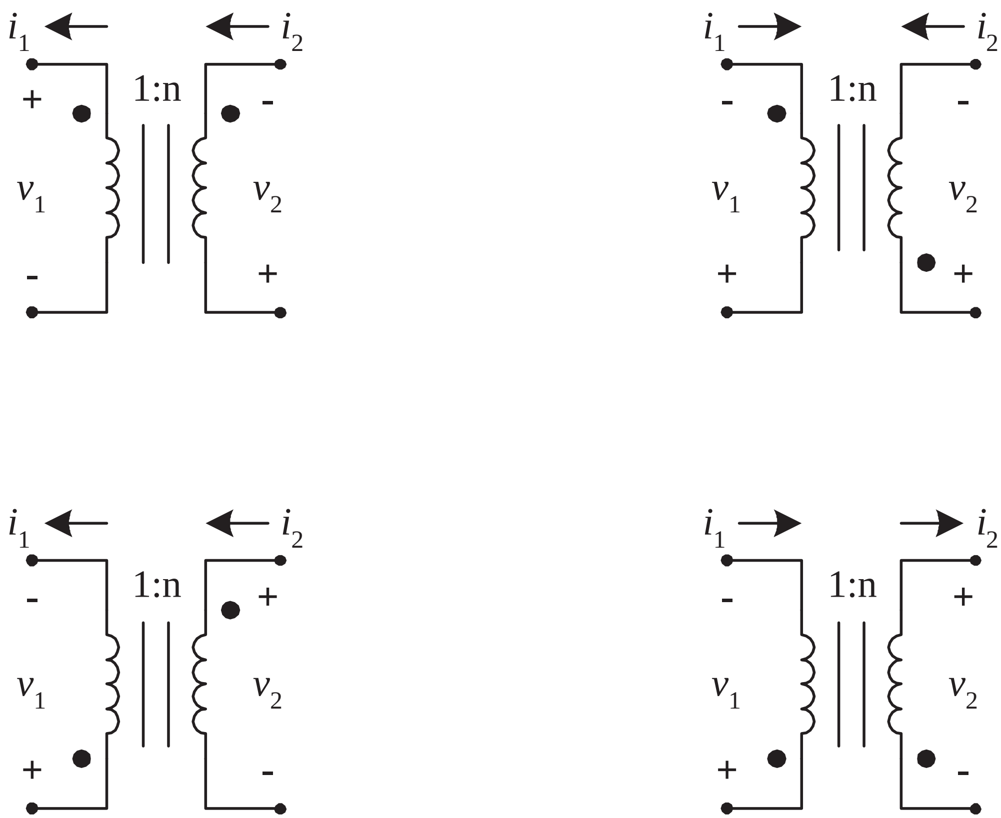
- The dot convention determines the polarity of the defining equations
- **Dot Convention**: If v_1 and v_2 are both positive or both negative at the dotted terminals, use $+n$. Otherwise, use -n.
- If i_1 and i_2 both enter or both leave the dotted terminals, use -n. Otherwise, use $+n$.

Example 6: The Dot Convention for Ideal Transformers



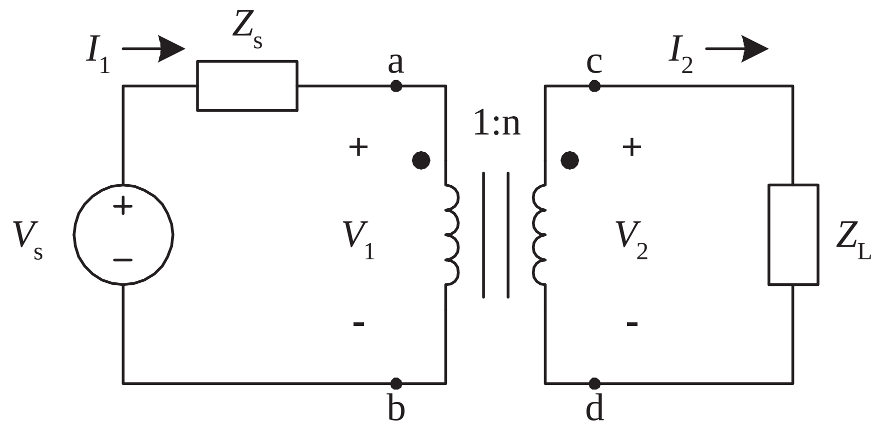
Write the defining equations for each of the circuits shown above.

Example 7: The Dot Convention for Ideal Transformers



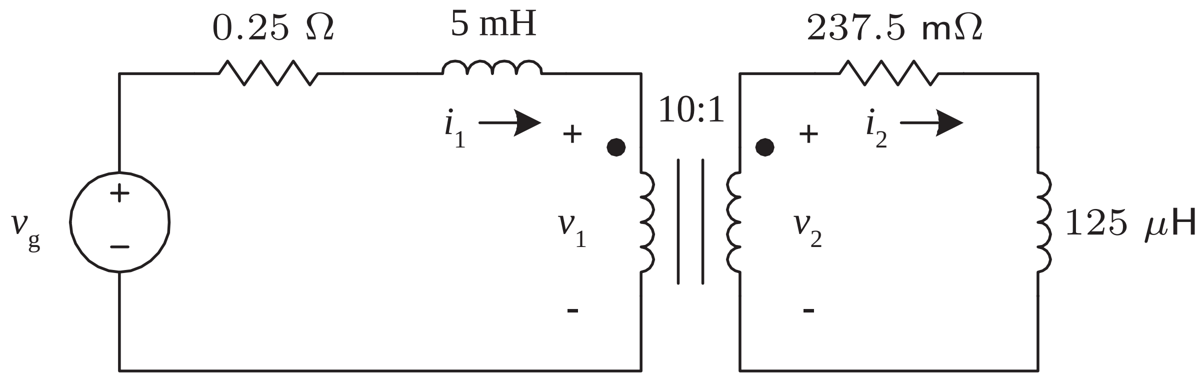
Write the defining equations for each of the circuits shown above.

Ideal Transformer: Reflected Impedance



$$Z_R = \frac{V_1}{I_1} = \frac{V_2 \frac{1}{n}}{I_2 n} = \frac{V_2}{I_2}\frac{1}{n^2} = \frac{Z_L}{n^2}$$

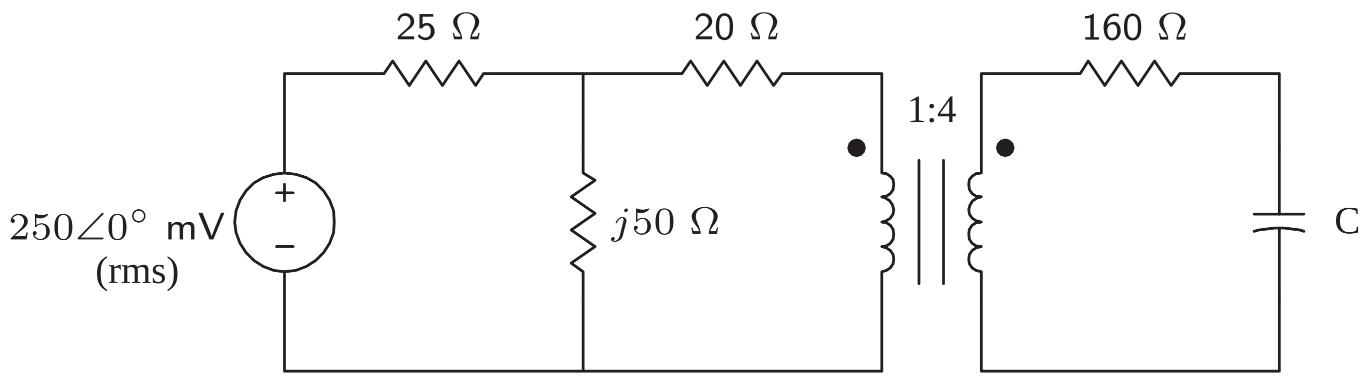
Example 8: Ideal Transformers



If $v_g = 2500\cos(400t)$ V, find i_1, v_1, i_2, and v_2.

Example 8: Workspace

Example 9: Ideal Transformers



Find the value of C that maximizes the power absorbed by the $160\ \Omega$ resistor. What is the average power delivered for this value of C? Replace the resistor with a variable resistor and find the value that maximizes the power delivered? What is the maximum average power that can be delivered?

Example 9: Workspace (1)

Example 9: Workspace (2)