Example 14.12(c).

14.6 COMPLEX NUMBERS

In our analysis of dc networks, we found it necessary to determine the algebraic sum of voltages and currents. Since the same will also be true for ac networks, the question arises, How do we determine the algebraic sum of two or more voltages (or currents) that are varying sinusoidally? Although one solution would be to find the algebraic sum on a point-to-point basis (as shown in Section 14.12), this would be a long and tedious process in which accuracy would be directly related to the scale employed.

It is the purpose of this chapter to introduce a system of **complex numbers** that, when related to the sinusoidal ac waveform, will result in a technique for finding the algebraic sum of sinusoidal waveforms that is quick, direct, and accurate. In the following chapters, the technique will be extended to permit the analysis of sinusoidal ac networks in a manner very similar to that applied to dc networks. The methods and theorems as described for dc networks can then be applied to sinusoidal ac networks with little difficulty.

A **complex number** represents a point in a two-dimensional plane located with reference to two distinct axes. This point can also determine a radius vector drawn from the origin to the point. The horizontal axis is called the *real* axis, while the vertical axis is called the *imaginary* axis. Both are labeled in Fig. 14.35. Every number from zero to $\pm\infty$ can be represented by some point along the real axis. Prior to the development of this system of complex numbers, it was believed that

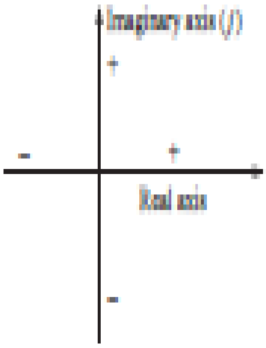

FIG. 14.35

Defining the real and imaginary axes of a complex plane.

14.7 RECTANGULAR FORM

The format for the rectangular form is

$$\mathbf{C} = X + jY \tag{14.20}$$

as shown in Fig. 14.36. The letter **C** was chosen from the word "complex." The boldface notation is for any number with magnitude and direction. The *italic* is for magnitude only.

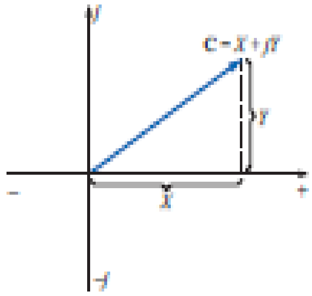

FIG. 14.36
Defining the rectangular form.

EXAMPLE 14.13 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 3 + j4$
b. $\mathbf{C} = 0 - j6$
c. $\mathbf{C} = -10 - j20$

Solutions:

a. See Fig. 14.37.
b. See Fig. 14.38.
c. See Fig. 14.39.

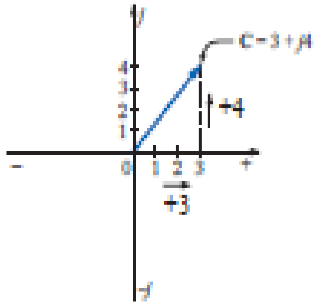

FIG. 14.37
Example 14.13(a).

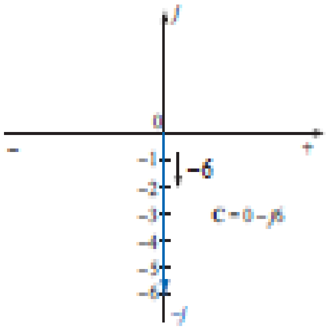

FIG. 14.38
Example 14.13(b).

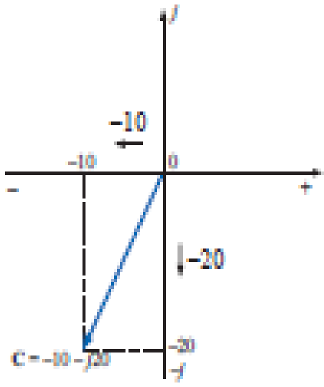

FIG. 14.39
Example 14.13(c).

14.8 POLAR FORM

The format for the polar form is

$$\mathbf{C} = Z \angle \theta \tag{14.21}$$

with the letter Z chosen from the sequence X, Y, Z.

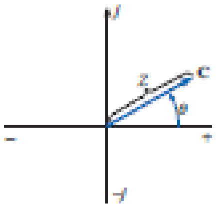

FIG. 14.40
Defining the polar form.

where Z indicates magnitude only and θ is always measured counterclockwise (CCW) from the *positive real axis*, as shown in Fig. 14.40. Angles measured in the clockwise direction from the positive real axis must have a negative sign associated with them.

A negative sign in front of the polar form has the effect shown in Fig. 14.41. Note that it results in a complex number directly opposite the complex number with a positive sign.

$$-\mathbf{C} = -Z\angle\theta = Z\angle\theta \pm 180° \qquad (14.22)$$

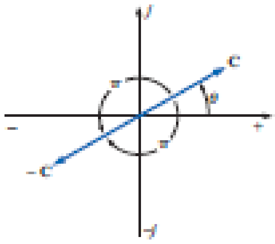

FIG. 14.41
Demonstrating the effect of a negative sign on the polar form.

EXAMPLE 14.14 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 5\angle 30°$
b. $\mathbf{C} = 7\angle -120°$
c. $\mathbf{C} = -4.2\angle 60°$

Solutions:

a. See Fig. 14.42.
b. See Fig. 14.43.
c. See Fig. 14.44.

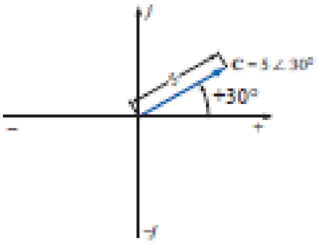

FIG. 14.42
Example 14.14(a).

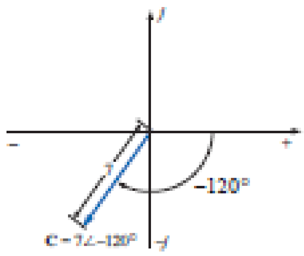

FIG. 14.43
Example 14.14(b).

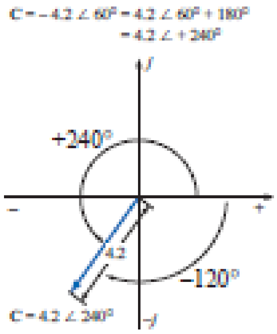

FIG. 14.44
Example 14.14(c).

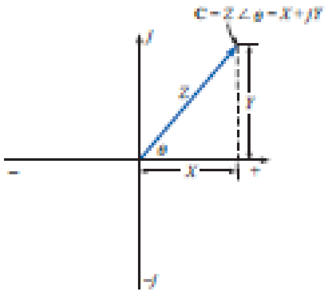

FIG. 14.45
Conversion between forms.

14.9 CONVERSION BETWEEN FORMS

The two forms are related by the following equations, as illustrated in Fig. 14.45.

Rectangular to Polar

$$Z = \sqrt{X^2 + Y^2} \qquad (14.23)$$

$$\theta = \tan^{-1}\frac{Y}{X} \qquad (14.24)$$

Polar to Rectangular

$$X = Z \cos \theta \tag{14.25}$$

$$Y = Z \sin \theta \tag{14.26}$$

EXAMPLE 14.15 Convert the following from rectangular to polar form:

$$\mathbf{C} = 3 + j\,4 \qquad \text{(Fig. 14.46)}$$

Solution:

$$Z = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

and $$\mathbf{C} = \mathbf{5 \angle 53.13^\circ}$$

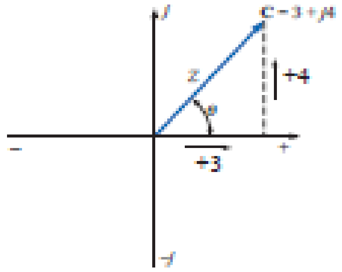

FIG. 14.46
Example 14.15.

EXAMPLE 14.16 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10 \angle 45^\circ \qquad \text{(Fig. 14.47)}$$

Solution:

$$X = 10 \cos 45^\circ = (10)(0.707) = 7.07$$
$$Y = 10 \sin 45^\circ = (10)(0.707) = 7.07$$

and $$\mathbf{C = 7.07 + j7.07}$$

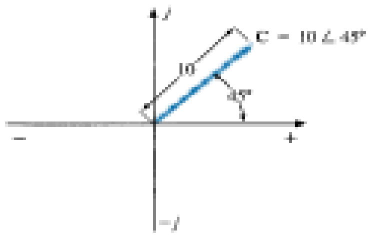

FIG. 14.47
Example 14.16.

If the complex number should appear in the second, third, or fourth quadrant, simply convert it in that quadrant, and carefully determine the proper angle to be associated with the magnitude of the vector.

EXAMPLE 14.17 Convert the following from rectangular to polar form:

$$\mathbf{C} = -6 + j\,3 \qquad \text{(Fig. 14.48)}$$

Solution:

$$Z = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 6.71$$

$$\beta = \tan^{-1}\left(\frac{3}{6}\right) = 26.57^\circ$$

$$\theta = 180^\circ - 26.57^\circ = 153.43^\circ$$

and $$\mathbf{C} = \mathbf{6.71 \angle 153.43^\circ}$$

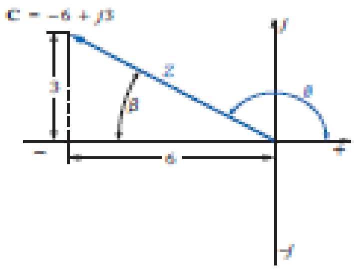

FIG. 14.48
Example 14.17.

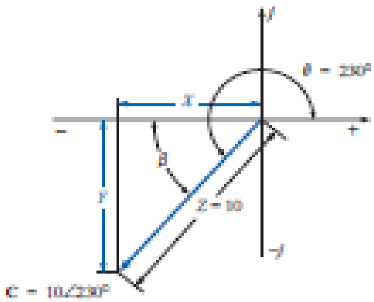

FIG. 14.49
Example 14.18.

EXAMPLE 14.18 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10 \angle 230° \qquad \text{(Fig. 14.49)}$$

Solution:

$$X = Z\cos\beta = 10\cos(230° - 180°) = 10\cos 50°$$
$$= (10)(0.6428) = 6.428$$
$$Y = Z\sin\beta = 10\sin 50° = (10)(0.7660) = 7.660$$

and $$\mathbf{C} = \mathbf{-6.428 - j7.660}$$

14.10 MATHEMATICAL OPERATIONS WITH COMPLEX NUMBERS

Complex numbers lend themselves readily to the basic mathematical operations of addition, subtraction, multiplication, and division. A few basic rules and definitions must be understood before considering these operations.

Let us first examine the symbol j associated with imaginary numbers. By definition,

$$j = \sqrt{-1} \qquad (14.27)$$

Thus, $$j^2 = -1 \qquad (14.28)$$

and $$j^3 = j^2 j = -1j = -j$$

with $$j^4 = j^2 j^2 = (-1)(-1) = +1$$
$$j^5 = j$$

and so on. Further,

$$\frac{1}{j} = (1)\left(\frac{1}{j}\right) = \left(\frac{j}{j}\right)\left(\frac{1}{j}\right) = \frac{j}{j^2} = \frac{j}{-1}$$

and $$\frac{1}{j} = -j \qquad (14.29)$$

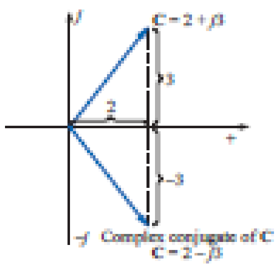

FIG. 14.50
Defining the complex conjugate of a complex number in rectangular form.

Complex Conjugate

The **conjugate** or **complex conjugate** of a complex number can be found by simply changing the sign of the imaginary part in the rectangular form or by using the negative of the angle of the polar form. For example, the conjugate of

$$\mathbf{C} = 2 + j3$$

is $$2 - j3$$

as shown in Fig. 14.50. The conjugate of

$$\mathbf{C} = 2 \angle 30°$$

is $$2 \angle -30°$$

as shown in Fig. 14.51.

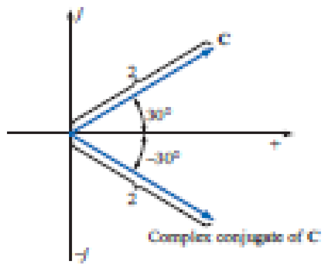

FIG. 14.51
Defining the complex conjugate of a complex number in polar form.

Reciprocal

The reciprocal of a complex number is 1 divided by the complex number. For example, the reciprocal of

$$\mathbf{C} = X + jY$$

is

$$\frac{1}{X + jY}$$

and of $Z \angle \theta$,

$$\frac{1}{Z \angle \theta}$$

We are now prepared to consider the four basic operations of *addition, subtraction, multiplication,* and *division* with complex numbers.

Addition

To add two or more complex numbers, simply add the real and imaginary parts separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\boxed{\mathbf{C}_1 + \mathbf{C}_2 = (\pm X_1 \pm X_2) + j(\pm Y_1 \pm Y_2)} \qquad (14.30)$$

There is really no need to memorize the equation. Simply set one above the other and consider the real and imaginary parts separately, as shown in Example 14.19.

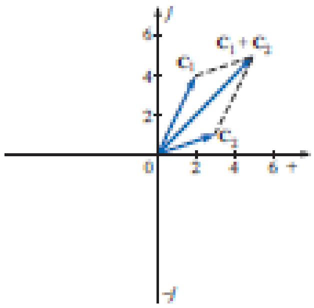

FIG. 14.52
Example 14.19(a).

EXAMPLE 14.19

a. Add $\mathbf{C}_1 = 2 + j4$ and $\mathbf{C}_2 = 3 + j1$.
b. Add $\mathbf{C}_1 = 3 + j6$ and $\mathbf{C}_2 = -6 + j3$.

Solutions:

a. By Eq. (14.30),

$$\mathbf{C}_1 + \mathbf{C}_2 = (2 + 3) + j(4 + 1) = 5 + j5$$

Note Fig. 14.52. An alternative method is

$$\begin{array}{r} 2 + j4 \\ \underline{3 + j1} \\ \downarrow \quad \downarrow \\ 5 + j5 \end{array}$$

b. By Eq. (14.30),

$$\mathbf{C}_1 + \mathbf{C}_2 = (3 - 6) + j(6 + 3) = -3 + j9$$

Note Fig. 14.53. An alternative method is

$$\begin{array}{r} 3 + j6 \\ \underline{-6 + j3} \\ \downarrow \quad \downarrow \\ -3 + j9 \end{array}$$

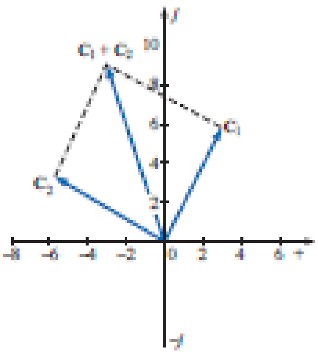

FIG. 14.53
Example 14.19(b).