14.12 PHASORS

As noted earlier in this chapter, the addition of sinusoidal voltages and currents will frequently be required in the analysis of ac circuits. One lengthy but valid method of performing this operation is to place both sinusoidal waveforms on the same set of axes and add algebraically the magnitudes of each at every point along the abscissa, as shown for $c = a + b$ in Fig. 14.62. This, however, can be a long and tedious process with limited accuracy. A shorter method uses the rotating radius vector first appearing in Fig. 13.16. This *radius vector*, having a *constant magnitude* (length) with *one end fixed at the origin*, is called a **phasor** when applied to electric circuits. During its rotational development of the sine wave, the phasor will, at the instant $t = 0$, have the positions shown in Fig. 14.63(a) for each waveform in Fig. 14.63(b).

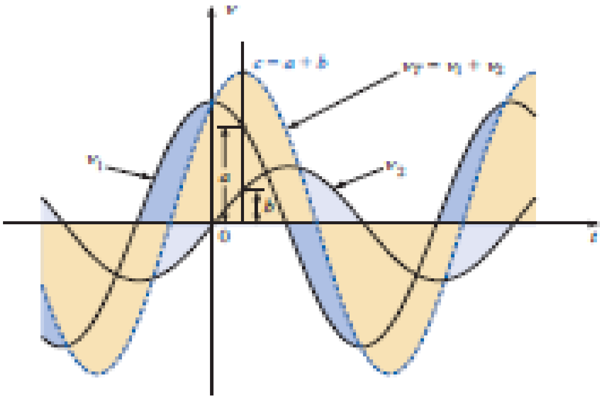

FIG. 14.62
Adding two sinusoidal waveforms on a point-by-point basis.

Note in Fig. 14.63(b) that v_2 passes through the horizontal axis at $t = 0$ s, requiring that the radius vector in Fig. 14.63(a) be on the horizontal axis to ensure a vertical projection of zero volts at $t = 0$ s. Its length in Fig. 14.63(a) is equal to the peak value of the sinusoid as required by the radius vector of Fig. 13.16. The other sinusoid has passed through 90° of its rotation by the time $t = 0$ s is reached and

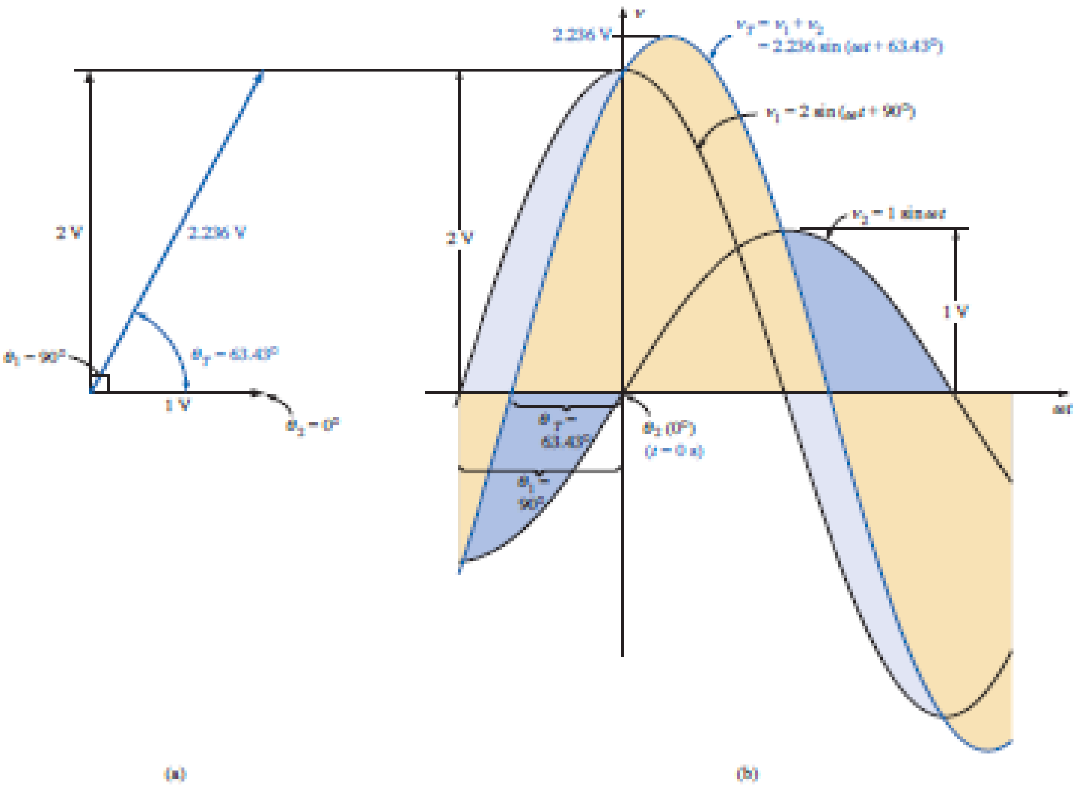

FIG. 14.63

(a) The phasor representation of the sinusoidal waveforms of Fig. 14.63(b); (b) finding the sum of two sinusoidal waveforms of v_1 and v_2.

therefore has its maximum vertical projection as shown in Fig. 14.63(a). Since the vertical projection is a maximum, the peak value of the sinusoid that it will generate is also attained at $t = 0$ s, as shown in Fig. 14.63(b). Note also that $v_T = v_1$ at $t = 0$ s since $v_2 = 0$ V at this instant.

It can be shown [see Fig. 14.63(a)] using the vector algebra described in Section 14.10 that

$$1 \text{ V} \angle 0° + 2 \text{ V} \angle 90° = 2.236 \text{ V} \angle 63.43°$$

In other words, if we convert v_1 and v_2 to the phasor form using

$$v = V_m \sin(\omega t \pm \theta) \Rightarrow V_m \angle \pm\theta$$

and add them using complex number algebra, we can find the phasor form for v_T with very little difficulty. It can then be converted to the time domain and plotted on the same set of axes, as shown in Fig. 14.63(b). Figure 14.63(a), showing the magnitudes and relative positions of the various phasors, is called a **phasor diagram**. It is actually a "snapshot" of the rotating radius vectors at $t = 0$ s.

In the future, therefore, if the addition of two sinusoids is required, they should first be converted to the phasor domain and the sum found using complex algebra. The result can then be converted to the time domain.

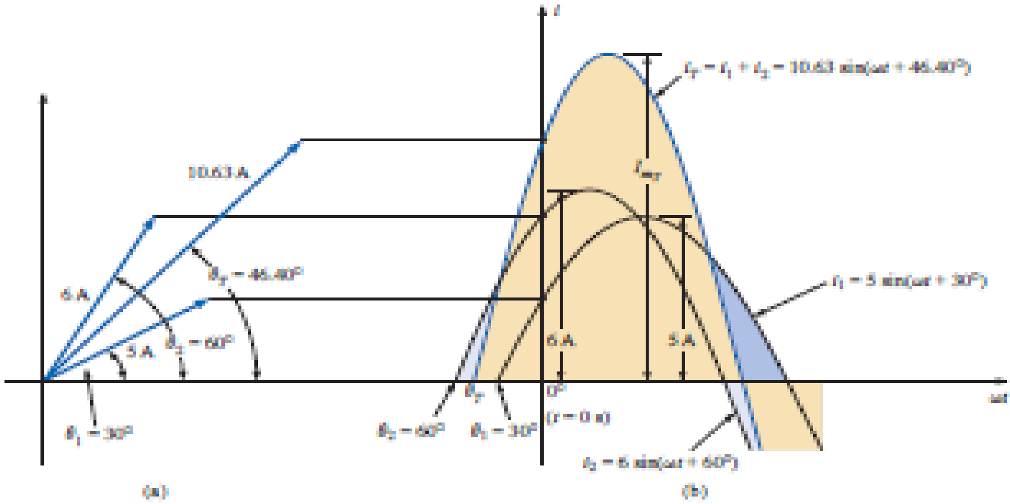

FIG. 14.64
Adding two sinusoidal currents with phase angles other than 90°.

The case of two sinusoidal functions having phase angles different from 0° and 90° appears in Fig. 14.64. Note again that the vertical height of the functions in Fig. 14.64(b) at $t = 0$ s is determined by the rotational positions of the radius vectors in Fig. 14.64(a).

Since the rms, rather than the peak, values are used almost exclusively in the analysis of ac circuits, the phasor will now be redefined for the purposes of practicality and uniformity as having a magnitude equal to the *rms value* of the sine wave it represents. The angle associated with the phasor will remain as previously described—the phase angle.

In general, for all of the analyses to follow, the phasor form of a sinusoidal voltage or current will be

$$\mathbf{V} = V \angle \theta \quad \text{and} \quad \mathbf{I} = I \angle \theta$$

where V and I are rms values and θ is the phase angle. It should be pointed out that in phasor notation, the sine wave is always the reference, and the frequency is not represented.

Phasor algebra for sinusoidal quantities is applicable only for waveforms having the same frequency.

EXAMPLE 14.29 Convert the following from the time to the phasor domain:

Time Domain	Phasor Domain
a. $\sqrt{2}(50) \sin \omega t$	$\mathbf{50 \angle 0°}$
b. $69.6 \sin(\omega t + 72°)$	$(0.707)(69.6) \angle 72° = \mathbf{49.21 \angle 72°}$
c. $45 \cos \omega t$	$(0.707)(45) \angle 90° = \mathbf{31.82 \angle 90°}$

EXAMPLE 14.30 Write the sinusoidal expression for the following phasors if the frequency is 60 Hz:

Phasor Domain	Time Domain
a. $\mathbf{I} = 10 \angle 30°$	$i = \sqrt{2}(10)\sin(2\pi 60t + 30°)$
	and $i = \mathbf{14.14 \sin(377t + 30°)}$
b. $\mathbf{V} = 115 \angle -70°$	$v = \sqrt{2}(115)\sin(377t - 70°)$
	and $v = \mathbf{162.6 \sin(377t - 70°)}$

EXAMPLE 14.31 Find the input voltage of the circuit of Fig. 14.65 if

$$\left.\begin{aligned} v_a &= 50\sin(377t + 30°) \\ v_b &= 30\sin(377t + 60°) \end{aligned}\right\} f = 60\text{ Hz}$$

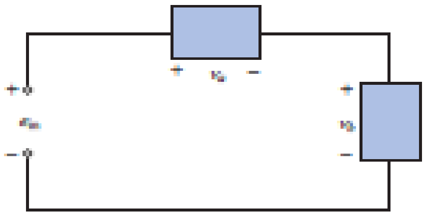

FIG. 14.65
Example 14.31.

Solution: Applying Kirchhoff's voltage law, we have

$$e_{in} = v_a + v_b$$

Converting from the time to the phasor domain yields

$$v_a = 50\sin(377t + 30°) \Rightarrow \mathbf{V}_a = 35.35\text{ V} \angle 30°$$
$$v_b = 30\sin(377t + 60°) \Rightarrow \mathbf{V}_b = 21.21\text{ V} \angle 60°$$

Converting from polar to rectangular form for addition yields

$$\mathbf{V}_a = 35.35\text{ V} \angle 30° = 30.61\text{ V} + j\,17.68\text{ V}$$
$$\mathbf{V}_b = 21.21\text{ V} \angle 60° = 10.61\text{ V} + j\,18.37\text{ V}$$

Then

$$\begin{aligned}\mathbf{E}_{in} = \mathbf{V}_a + \mathbf{V}_b &= (30.61\text{ V} + j\,17.68\text{ V}) + (10.61\text{ V} + j\,18.37\text{ V}) \\ &= 41.22\text{ V} + j\,36.05\text{ V}\end{aligned}$$

Converting from rectangular to polar form, we have

$$\mathbf{E}_{in} = 41.22\text{ V} + j\,36.05\text{ V} = 54.76\text{ V} \angle 41.17°$$

Converting from the phasor to the time domain, we obtain

$$\mathbf{E}_{in} = 54.76\text{ V} \angle 41.17° \Rightarrow e_{in} = \sqrt{2}(54.76)\sin(377t + 41.17°)$$

and $$e_{in} = \mathbf{77.43 \sin(377t + 41.17°)}$$

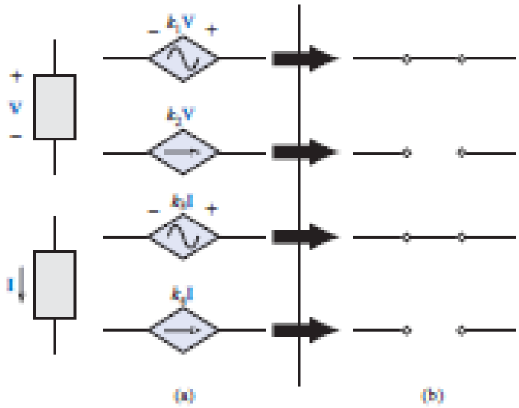

FIG. 17.4
Conditions of V = 0 V and I = 0 A for a controlled source.

17.3 SOURCE CONVERSIONS

When applying the methods to be discussed, it may be necessary to convert a current source to a voltage source, or a voltage source to a current source. This source conversion can be accomplished in much the same manner as for dc circuits, except now we shall be dealing with phasors and impedances instead of just real numbers and resistors.

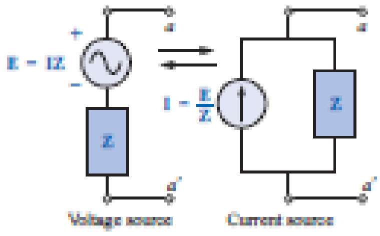

FIG. 17.5
Source conversion.

Independent Sources

In general, the format for converting one type of independent source to another is as shown in Fig. 17.5.

EXAMPLE 17.1 Convert the voltage source of Fig. 17.6(a) to a current source.

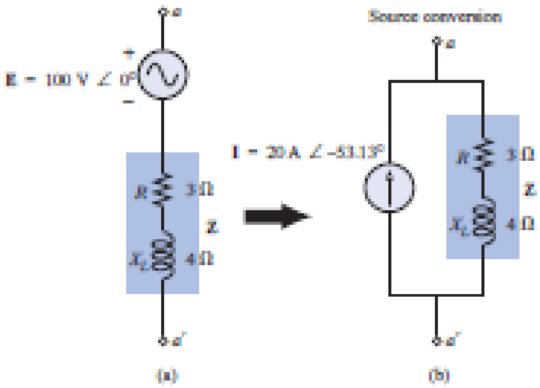

FIG. 17.6
Example 17.1.

Solution:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{100\text{ V} \angle 0°}{5\ \Omega \angle 53.13°}$$
$$= \mathbf{20\ A \angle -53.13°} \qquad \text{[Fig. 17.6(b)]}$$

EXAMPLE 17.2 Convert the current source of Fig. 17.7(a) to a voltage source.

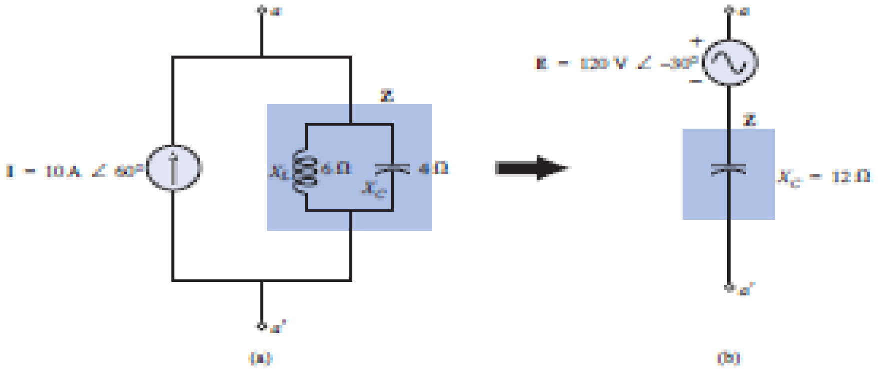

FIG. 17.7
Example 17.2.

Solution:

$$\mathbf{Z} = \frac{\mathbf{Z}_C\mathbf{Z}_L}{\mathbf{Z}_C + \mathbf{Z}_L} = \frac{(X_C \angle -90°)(X_L \angle 90°)}{-j\,X_C + j\,X_L}$$
$$= \frac{(4\ \Omega \angle -90°)(6\ \Omega \angle 90°)}{-j\,4\ \Omega + j\,6\ \Omega} = \frac{24\ \Omega \angle 0°}{2 \angle 90°}$$
$$= \mathbf{12\ \Omega \angle -90°} \qquad \text{[Fig. 17.7(b)]}$$
$$\mathbf{E} = \mathbf{IZ} = (10\text{ A} \angle 60°)(12\ \Omega \angle -90°)$$
$$= \mathbf{120\ V \angle -30°} \qquad \text{[Fig. 17.7(b)]}$$

Dependent Sources

For dependent sources, the direct conversion of Fig. 17.5 can be applied if the controlling variable (**V** or **I** in Fig. 17.4) is not determined by a portion of the network to which the conversion is to be applied. For example, in Figs. 17.8 and 17.9, **V** and **I**, respectively, are controlled by an external portion of the network. Conversions of the other kind, where **V** and **I** are controlled by a portion of the network to be converted, will be considered in Sections 18.3 and 18.4.

EXAMPLE 17.3 Convert the voltage source of Fig. 17.8(a) to a current source.

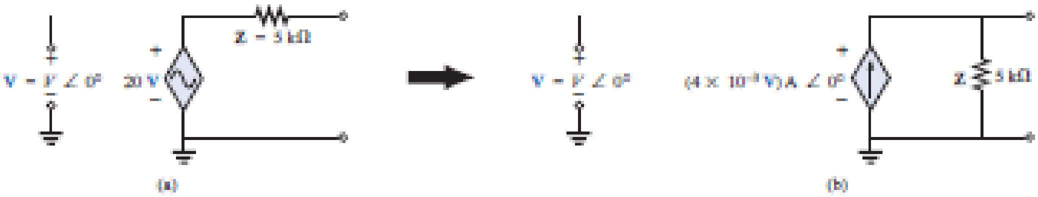

FIG. 17.8
Source conversion with a voltage-controlled voltage source.

Solution:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{(20V)\ V \angle 0^\circ}{5\ k\Omega \angle 0^\circ}$$
$$= (4 \times 10^{-3}\ V)\ A \angle 0^\circ \qquad \text{[Fig. 17.8(b)]}$$

EXAMPLE 17.4 Convert the current source of Fig. 17.9(a) to a voltage source.

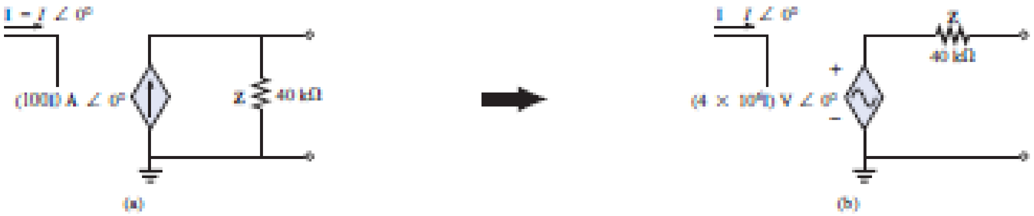

FIG. 17.9
Source conversion with a current-controlled current source.

Solution:

$$\mathbf{E} = \mathbf{IZ} = [(100\mathbf{I})\ A \angle 0^\circ][40\ k\Omega \angle 0^\circ]$$
$$= (4 \times 10^6\mathbf{I})\ V \angle 0^\circ \qquad \text{[Fig. 17.9(b)]}$$

17.4 MESH ANALYSIS

General Approach

Independent Voltage Sources Before examining the application of the method to ac networks, the student should first review the appropriate sections on mesh analysis in Chapter 8 since the content of this section will be limited to the general conclusions of Chapter 8.

The general approach to mesh analysis for independent sources includes the same sequence of steps appearing in Chapter 8. In fact, throughout this section the only change from the dc coverage will be to substitute impedance for resistance and admittance for conductance in the general procedure.

described shortly must be applied.

EXAMPLE 17.5 Using the general approach to mesh analysis, find the current I_1 in Fig. 17.10.

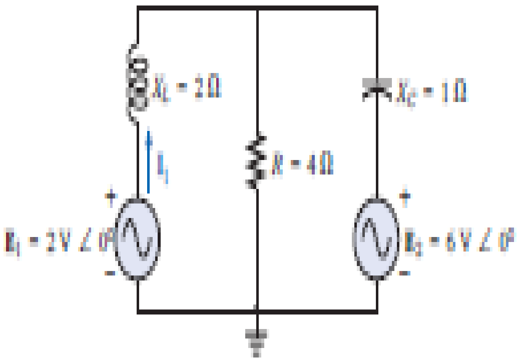

FIG. 17.10
Example 17.5.

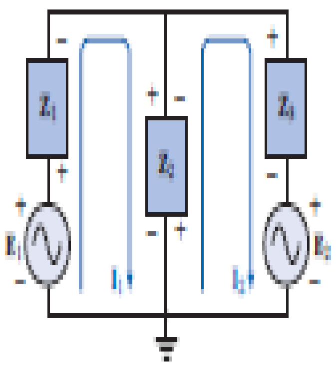

FIG. 17.11
Assigning the mesh currents and subscripted impedances for the network of Fig. 17.10.

Solution: When applying these methods to ac circuits, it is good practice to represent the resistors and reactances (or combinations thereof) by subscripted impedances. When the total solution is found in terms of these subscripted impedances, the numerical values can be substituted to find the unknown quantities.

The network is redrawn in Fig. 17.11 with subscripted impedances:

$$Z_1 = +jX_L = +j\,2\ \Omega \qquad E_1 = 2\ V \angle 0°$$
$$Z_2 = R = 4\ \Omega \qquad E_2 = 6\ V \angle 0°$$
$$Z_3 = -jX_C = -j\,1\ \Omega$$

Steps 1 and 2 are as indicated in Fig. 17.11.

Step 3:

$$+\mathbf{E}_1 - \mathbf{I}_1\mathbf{Z}_1 - \mathbf{Z}_2(\mathbf{I}_1 - \mathbf{I}_2) = 0$$
$$-\mathbf{Z}_2(\mathbf{I}_2 - \mathbf{I}_1) - \mathbf{I}_2\mathbf{Z}_3 - \mathbf{E}_2 = 0$$

or

$$\mathbf{E}_1 - \mathbf{I}_1\mathbf{Z}_1 - \mathbf{I}_1\mathbf{Z}_2 + \mathbf{I}_2\mathbf{Z}_2 = 0$$
$$-\mathbf{I}_2\mathbf{Z}_2 + \mathbf{I}_1\mathbf{Z}_2 - \mathbf{I}_2\mathbf{Z}_3 - \mathbf{E}_2 = 0$$

so that

$$\mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 = \mathbf{E}_1$$
$$\mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{I}_1\mathbf{Z}_2 = -\mathbf{E}_2$$

which are rewritten as

$$\begin{aligned} \mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 \qquad\qquad &= \mathbf{E}_1 \\ -\mathbf{I}_1\mathbf{Z}_2 \qquad + \mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) &= -\mathbf{E}_2 \end{aligned}$$

Step 4: Using determinants, we obtain

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{E}_1 & -\mathbf{Z}_2 \\ -\mathbf{E}_2 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_2 & -\mathbf{Z}_2 \\ -\mathbf{Z}_2 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{vmatrix}}$$
$$= \frac{\mathbf{E}_1(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{E}_2(\mathbf{Z}_2)}{(\mathbf{Z}_1 + \mathbf{Z}_2)(\mathbf{Z}_2 + \mathbf{Z}_3) - (\mathbf{Z}_2)^2}$$
$$= \frac{(\mathbf{E}_1 - \mathbf{E}_2)\mathbf{Z}_2 + \mathbf{E}_1\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}$$

Substituting numerical values yields

$$\mathbf{I}_1 = \frac{(2\text{ V} - 6\text{ V})(4\ \Omega) + (2\text{ V})(-j\,1\ \Omega)}{(+j\,2\ \Omega)(4\ \Omega) + (+j\,2\ \Omega)(-j\,2\ \Omega) + (4\ \Omega)(-j\,2\ \Omega)}$$
$$= \frac{-16 - j\,2}{j\,8 - j^2 2 - j\,4} = \frac{-16 - j\,2}{2 + j\,4} = \frac{16.12\text{ A}\angle -172.87^\circ}{4.47\angle 63.43^\circ}$$
$$= \mathbf{3.61\text{ A}\angle -236.30^\circ} \quad \text{or} \quad \mathbf{3.61\text{ A}\angle 123.70^\circ}$$

Dependent Voltage Sources For dependent voltage sources, the procedure is modified as follows:

1. Steps 1 and 2 are the same as those applied for independent voltage sources.
2. Step 3 is modified as follows: Treat each dependent source like an independent source when Kirchhoff's voltage law is applied to each independent loop. However, once the equation is written, substitute the equation for the controlling quantity to ensure that the unknowns are limited solely to the chosen mesh currents.
3. Step 4 is as before.

EXAMPLE 17.6 Write the mesh currents for the network of Fig. 17.12 having a dependent voltage source.

Solution:

Steps 1 and 2 are defined on Fig. 17.12.

Step 3: $\mathbf{E}_1 - \mathbf{I}_1 R_1 - R_2(\mathbf{I}_1 - \mathbf{I}_2) = 0$

$R_2(\mathbf{I}_2 - \mathbf{I}_1) + \mu\mathbf{V}_x - \mathbf{I}_2 R_3 = 0$

Then substitute $\mathbf{V}_x = (\mathbf{I}_1 - \mathbf{I}_2)R_2$

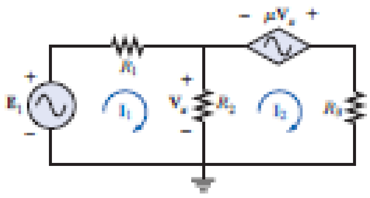

FIG. 17.12
Applying mesh analysis to a network with a voltage-controlled voltage source.