SIGNALS AND SYSTEMS

By Lecturer Ahmed Wael

Academic Year 2017 - 2018

LECTURE OUTLINES

- We need tools to build any communication system.
- Mathematics is our premium tool to do work with signals and systems.
- In this lecture we study:
  - Signals Classifications
  - Fourier Transform
  - Fourier Transform Properties
  - Energy Spectrum, Power Spectrum and Signal Bandwidth
  - Linear Systems and Signal Transmission

HOW DO WE COMMUNICATE ?!!

- Sound (Voice, Low Bit-Rate Underwater), Light (Optical Fiber, Heliograph), Electromagnetics Radiation (R.F. Signal)
- Baseband (e.g. Ethernet – 100 Mbps using Manchester Code), R.F. Modulation (e.g. Radio – 100 MHz Frequency Modulation)
- Analogue (e.g. AM / FM Radio), Digital (e.g. Music CD)
- Transmission Medium / Channel (e.g. Free Space, Optical Fiber, Coaxial Cable)

WHAT WE SUFFER FROM

- Noise / Interference (e.g. Atmospheric R.F. Noise, Thermal Noise in Electronic Components)
- How fast we communicate (> 10 Tbps)
- How do we cope with errors – detect and/or correct ?!!
- How can we have multiple access.

REVIEW – SIGNALS

- The most general class of signals used for modeling the interaction of signals in systems are sinusoids:

$$x(t) = A\cos(\omega_0 t + \Phi)$$

A, ω_0, Φ are Amplitude, angular frequency and phase, respectively

- <u>Example:</u>

$$x(t) = 10\cos[2\pi(440)t - 0.4\pi]$$

The pattern repeat itself every $\frac{1}{440} = 2.27\ ms$

REVIEW – SIGNALS

PROPERTY	EQUATION
Equivalence	$\sin\emptyset = \cos\left(\emptyset - \frac{\pi}{2}\right)$ or $\cos\emptyset = \sin(\emptyset + \frac{\pi}{2})$
Periodicity	$\cos(\emptyset - 2\pi k) = \cos\phi$
Evenness of cosine	$\cos(-\emptyset) = \cos\emptyset$
Oddness of sine	$\sin(-\emptyset) = -\sin\emptyset$
Trigonometric identity 1	$sin^2\emptyset + cos^2\emptyset = 1$
Trigonometric identity 2	$\cos 2\emptyset = s\sin\emptyset\cos\emptyset$
Trigonometric identity 3	$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$
Trigonometric identity 4	$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$
Trigonometric identity 5	$cos^2\emptyset = 0.5(1 + \cos 2\emptyset)$
Trigonometric identity 6	$sin^2\emptyset = 0.5(1 - \cos 2\emptyset)$

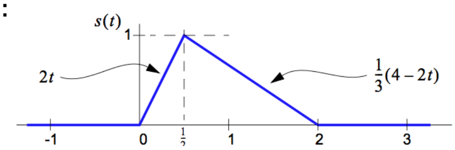
REVIEW – SIGNALS

- Phase and Time Shits of Sinusoidal Signals:
- We wish to consider the signal $x_1(t) = s(t-2)$.
- As a starting point we note that $s(t)$ is active over just the interval $0 \le t \le 2$, so with $t \to t-2$, we have:

$$0 \le (t-2) \le 2 \implies 2 \le t \le 4$$

which means that $x_1(t)$ is active over $2 \le t \le 4$.

$$s(t) = \begin{cases} 2t, & 0 \le t \le 1/2 \\ \frac{1}{3}(4-2t), & 1/2 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

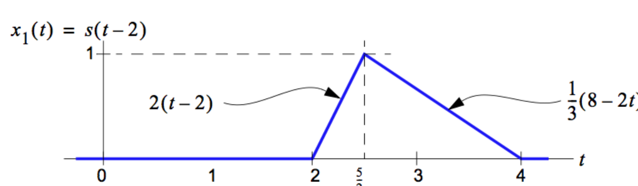


REVIEW – SIGNALS

$$x_1(t) = \begin{cases} 2(t-2), & 0 \le (t-2) \le 1/2 \\ \frac{1}{3}(4-2(t-2)), & 1/2 \le (t-2) \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2t-4, & 2 \le t \le 5/2 \\ \frac{1}{3}(8-2t), & 5/2 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}$$

We say that the signal is delayed in time to $s(t)$ if $t_1 > 0$, and advanced in time relative to $s(t)$ if $t_1 < 0$.



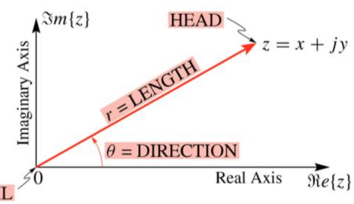
COMPLEX NUMBERS

- A complex number is another representation of numbers but in term of $z = (x, y)$.
- The 1st number, x, is called the real real and y is called the imaginary part.

$$z = (x, y) = x + jy$$

- The polar form is:

$$z = r\, e^{j\emptyset} = r\angle\,\emptyset = |z|e^{j\,\arg z}$$



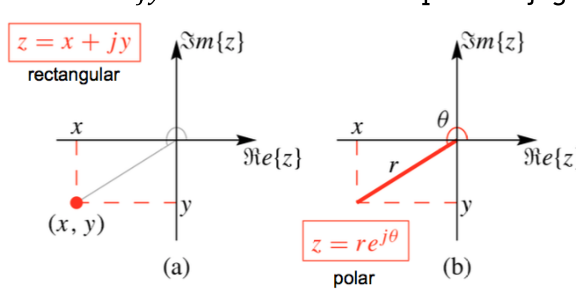
- Identities: let $z1 = x1 + jy1,\ z2 = x2 + jy2$
  - $z_T = (x1 + x2) + j(y1 + y1)$ (sum)
  - $z_T = (x1 - x2) + j(y1 - y1)$ (difference)
  - $z1\, z2 = (x1x2\ - y1y2) + j(x1y2 + y1x2)$ (Product)
  - $\frac{z1}{z2} = \frac{(x1x2+y1y2)-j(x1y2-y1x2)}{x2^2y2^2}$ (Division)

COMPLEX NUMBERS

$|z1| = \sqrt{x1^2 +\ y1^2}$ Magnitude of complex number

$\angle\, z1 =\ tan^{-1}\left(\frac{y1}{x1}\right)$ Angle of complex number

- $z1^* = x1\ - jy1$ Complex conjugate



COMPLEX EXPONENTIALS AND PHASORS

- Signal can be expressed in term of complex exponentials.
- The complex exponential form is written as:

$$z(t) = A\, e^{j(\omega_0 t + \phi)}$$

- Note that: $|z(t)| = A$ and $\angle z(t) = \arg\{z(t)\} = \omega_0 t + \phi$
- Euler's formula:

$$z(t) = Ae^{j(\omega_0 t + \phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$$

$$Re\{z(t)\} = A\cos(\omega_0 t + \emptyset) \quad \text{and} \quad Im\{z(t)\} = A\sin(\omega_0 t + \emptyset)$$

SIGNALS IN TIME DOMAIN

- A signal is a set of data or information, which can be represented as a function of *TIME s(t).*
- Deterministic signal is a signal whose physical description is known completely, either in a mathematical form or a graphical form
  - Signal Energy: $E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt$
  - Signal Power: $P_s = \lim_{T\to\infty} \int_{-\infty}^{+\infty} |s(t)|^2 dt$
- Signal Classification:
  - Continuous Time vs Discrete Time
  - Periodic vs. Aperiodic

SIGNALS IN FREQUENCY DOMAIN

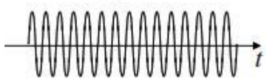
Time domain

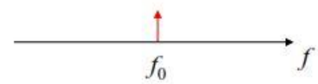
$s(t)$

Frequency domain

$S(f)=\int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$

- $\cos(2\pi f_0 t)$

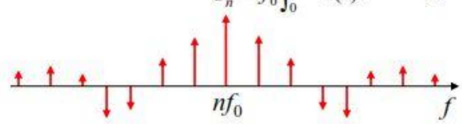




- Periodic signal with period $1/f_0$:

$s_n = f_0\int_0^{1/f_0} s(t)e^{-j2\pi n f_0 t}dt$

$\sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$



FOURIER ANALYSIS

- How do we represent the distribution of ENERGY / POWER of a signal with frequency?
  - A periodic waveform $x(t)$ (infinite energy / finite power) has a Fourier Series representation – power carried at discrete frequencies.
  - A non-periodic waveform $x(t)$ (finite energy / zero power) has a Fourier Transform representation – energy carried at ALL frequencies.
  - A 'random waveform' $x(t)$, or sample sequence from a random process (infinite energy / finite power), has a Power Spectral Density (PSD) representation, $S_x(f)$ – power carried at ALL frequencies.

FOURIER SERIES

- A continuous – time periodic function, $x(t)$, can (under certain conditions) be represented by ***FOURIER SERIES*** – i.e. the sum of different sinusoids/cosinusoids of different amplitudes, frequencies and phases.
- The Fourier series takes many forms:

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t) \qquad (1)$$

$$f(t) = A_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \phi_n) \qquad (2)$$

$$f(t) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{jn\omega_0 t} \qquad (3)$$

$$\alpha_n = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

This known as Fourier coefficient

FOURIER TRANSFORM

- Given a Time Domain signal $s(t)$, its Fourier Transform is defined as follow:

$$S(f) = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi ft}\, dt$$

Fourier Transform

- The Time Domain signal $s(t)$ can be expressed by $S(f)$ using an inverse FT:

$$s(t) = \int_{-\infty}^{+\infty} S(f) e^{j2\pi ft}\, df$$

Inverse Fourier Transform

- Fourier Spectrum of $s(t)$: $S(f)$
- Magnitude Spectrum of $s(t)$: $|\, S(f)\, |$

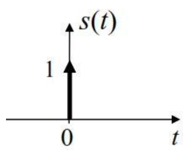
$$s(t) \Leftrightarrow S(f)$$

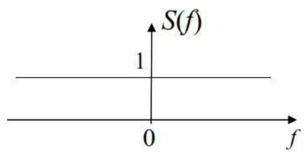
SPECTRUM OF UNIT IMPULSE

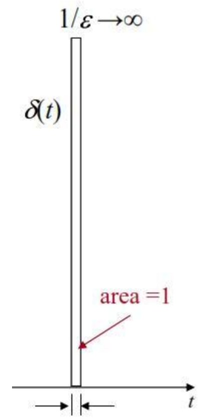
$\delta(t)$ is a unit impulse, which is zero everywhere except at $t=0$, and has unit area.

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

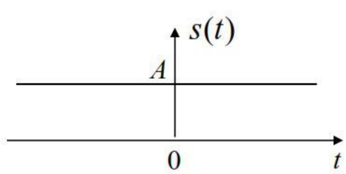
$$s(t) = \delta(t) \quad \Leftrightarrow \quad S(f) = \int_{-\infty} \delta(t)e^{-j2\pi ft}dt$$
$$= 1$$





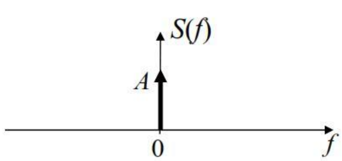


SPECTRUM OF CONSTANT SIGNAL



$$s(t) = A$$

$$S(f) = A\int_{-\infty}^{\infty} e^{-j2\pi ft}dt$$

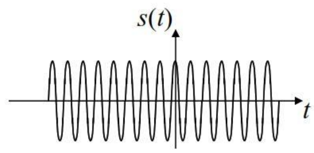


$$S(0) = A\int_{-\infty}^{\infty} e^{-j2\pi 0t}dt = A\int_{-\infty}^{\infty} 1dt = \infty$$

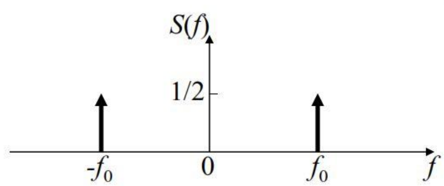
$$S(f) = A\int_{-\infty}^{\infty} e^{-j2\pi ft}dt = 0 \quad \text{for} \quad f \neq 0$$

$$S(f) = A\delta(f)$$

SPECTRUM OF SINUSOIDAL



$$s(t) = \cos(2\pi f_0 t)$$



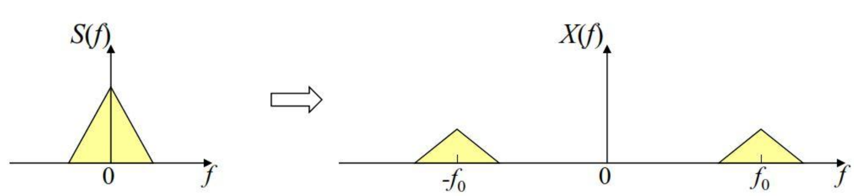
$$S(f) = \int_{-\infty}^{\infty} \cos 2\pi f_0 t \cdot e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} \tfrac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \cdot e^{-j2\pi ft} dt$$
$$= \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t} dt + \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t} dt$$
$$= \tfrac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$$

LOPE3202: Communication Systems 10/18/2017 19

SPECTRUM OF S(T)COS(2ΠFT)

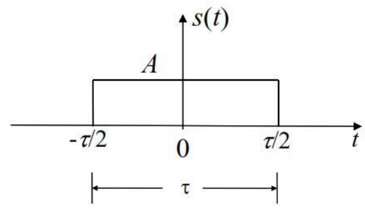
$$x(t) = s(t)\cos(2\pi f_0 t)$$

$$X(f) = \int_{-\infty}^{\infty} s(t)\cos(2\pi f_0 t) \cdot e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} s(t) \cdot \tfrac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \cdot e^{-j2\pi ft} dt$$
$$= \tfrac{1}{2} \int_{-\infty}^{\infty} s(t) \cdot e^{-j2\pi(f-f_0)t} dt + \tfrac{1}{2} \int_{-\infty}^{\infty} s(t) \cdot e^{-j2\pi(f+f_0)t} dt = \tfrac{1}{2}[S(f - f_0) + S(f + f_0)]$$

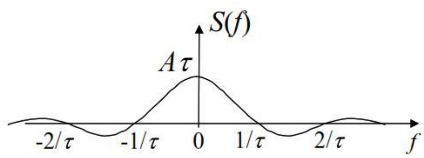


LOPE3202: Communication Systems 10/18/2017 20

SPECTRUM OF SINGLE RECTANGULAR PULSE



$$s(t) = \begin{cases} A & -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$



$$S(f) = A\int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}dt = A\cdot\frac{e^{-j\pi f\tau} - e^{+j\pi f\tau}}{-j2\pi f}$$

$$= A\tau\cdot\frac{\sin(\pi f\tau)}{\pi f\tau} = A\tau\,\mathrm{sinc}(f\tau)$$

$\mathrm{sinc}(x) \doteq \frac{\sin(\pi x)}{\pi x}$

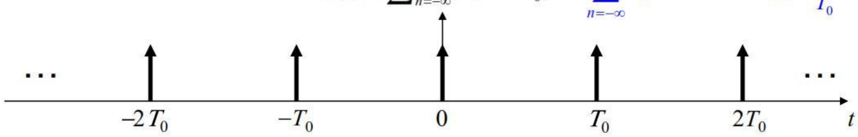
- sinc function is an even, oscillating function with a decreasing magnitude.
- It has unit peak at x=0, and zero crossing points at x= non-zero integers.

FOURIER TRANSFORM PROPERTIES

$\alpha s_1(t) + \beta s_2(t)$	$\Leftrightarrow$	$\alpha S_1(f) + \beta S_2(f)$	Linearity
$s_1(t)s_2(t)$	$\Leftrightarrow$	$S_1(f) * S_2(f)$	Convolution
$S(t)$	$\Leftrightarrow$	$s(-f)$	Duality
$s(t-\tau)$	$\Leftrightarrow$	$S(f)e^{-j2\pi f\tau}$	Time shift
$s(t)e^{-j2\pi f_0 t}$	$\Leftrightarrow$	$S(f+f_0)$	Frequency shift
$s(t)\cos(2\pi f_0 t)$	$\Leftrightarrow$	$\frac{1}{2}[S(f-f_0) + S(f+f_0)]$	Modulation
$s(at)$ (for any real $a \neq 0$)	$\Leftrightarrow$	$\frac{1}{\lvert a\rvert}S\left(\frac{f}{a}\right)$	Time scale

SPECTRUM OF IMPULSE TRAIN

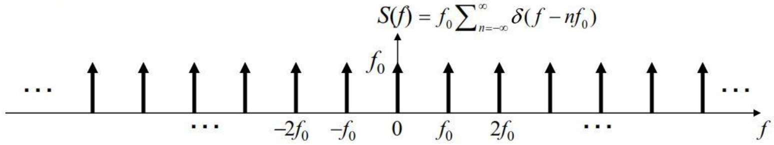
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t} \qquad f_0 = \frac{1}{T_0}$$



$$S(f) = \sum_{n=-\infty}^{\infty} s_n \delta(f - nf_0)$$

$$s_n = \frac{1}{T_0}\int_0^{T_0} s(t) e^{-j2\pi n f_0 t} dt = f_0 \int_0^{1/f_0} \delta(t) e^{-j2\pi n f_0 t} dt = f_0$$

$$S(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$



TIME AND FREQUENCY SIGNAL

- If the time-domain description of a signal is changed, then the frequency domain signal is changed in inverse manner. Hence, the arbitrary specifications of a signal cannot be performed in both time and frequency signal, but exclusively in one domain either time or frequency.
- If the ***FT*** of a signal in frequency domain has a finite band (zero outside the finite band), then the signal defined as ***strictly limited in frequency.*** The time domain in this case will be ***indefinitely.***
- If the ***IFT*** of a time domain signal has a finite band (zero outside the finite band), e.g. square pulse, then the frequency domain signal is extent to infinity. This type of signals known as ***strictly limited in time.***
- Signals can be **EITHER** band limited **OR** time limited.

ENERGY AND POWER TYPE SIGNALS

- Energy-Type Signal: A signal is an energy0type signal if and only if its energy is positive and finite

$s(t)$ is an energy-type signal if and only if $0 < E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt$

- Power-type Signal: A signal is a power type signal if and only if its power is positive and infinite

$s(t)$ is an energy-type signal if and only if $0 < P_s = \lim_{T\to\infty} \int_{-\infty}^{+\infty} |s(t)|^2 dt$

ENERGY AND ENERGY SPECTRUM

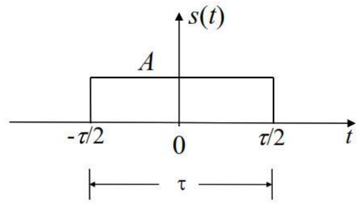
- Energy of energy-type signal:

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} s(t)s^*(t)dt = \int_{-\infty}^{\infty} s(t)\left[\int_{-\infty}^{\infty} S^*(f)e^{-j2\pi ft}df\right]dt$$
$$= \int_{-\infty}^{\infty} S^*(f)\left[\int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt\right]df = \int_{-\infty}^{\infty} S^*(f)S(f)df = \int_{-\infty}^{\infty} |S(f)|^2\, df$$
$$= \int_{-\infty}^{\infty} U_s(f)df$$

Parseval's Theorem: $E_s = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} |S(f)|^2\, df$

- Energy Spectrum: $U_s(f) \triangleq |S(f)|^2$

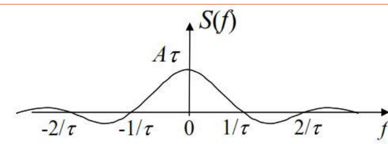
ENERGY SPECTRUM OF RECTANGULAR PULSE



$$s(t) = \begin{cases} A & -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

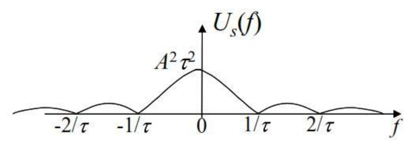
- Fourier spectrum:

$$S(f) = A\tau \text{sinc}(f\tau)$$



- Energy spectrum:

$$U_s(f) = |S(f)|^2 = A^2\tau^2 \text{sinc}^2(f\tau)$$



POWER AND POWER SPECTRUM

Power of power-type signal s(t):

$$s_T(t) \triangleq \begin{cases} s(t) & -T/2 < t < T/2 \\ 0 & otherwise \end{cases}$$

$$P_s = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |s(t)|^2\, dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |s_T(t)|^2\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |S_T(f)|^2\, df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} |S_T(f)|^2\, df = \int_{-\infty}^{\infty} G_s(f) df$$

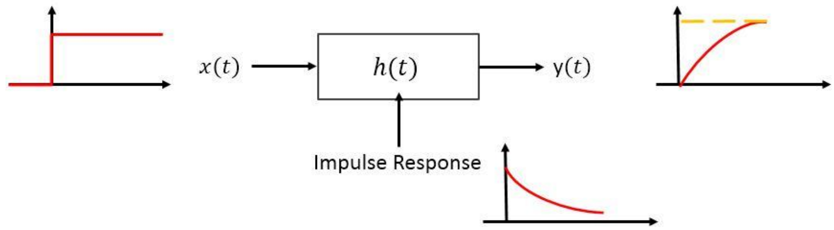
Power spectrum:

$$G_s(f) \triangleq \lim_{T\to\infty} \frac{1}{T} |S_T(f)|^2$$

$$G_s(f) \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} s(t+\tau)s^*(t)dt$$

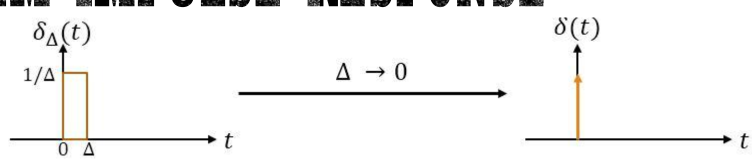
SIGNAL TRANSMISSION IN LINEAR SYSTEM

- **A System:** refers to any physical device that produces an output signal in response to an input signal. The input signal is called as an ***excitation*** and the output signal as a ***response.***



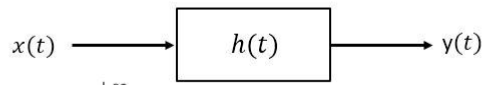
- Given impulse response, $h(t)$, and input signal, $x(t)$, how do we find $y(t)$?
- Any linear system is completely defined by knowledge of its impulse response, $h(t)$.
- So $h(t)$ is the response (output) of the system, when the input is an impulse – i.e. if $x(t) = \delta(t)$

SYSTEM IMPULSE RESPONSE



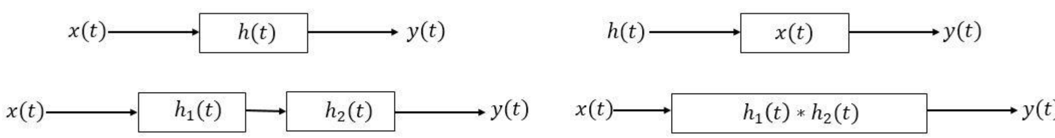
- Some basic properties of impulse response function:
  - $\int_{-\infty}^{\infty} A\delta(t - t_0)dt = A$
  - $\int_{-\infty}^{\infty} x(t)\,\delta(t - t_0)dt = x(t_0)$ Shifting property
  - $x(t)\delta(t - t_0)dt = x(t_0)\delta(t - t_0)dt$
  - $\delta(t) = \frac{du(t)}{dt}$ and $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$

THE CONVOLUTION INTEGRAL OF LTI



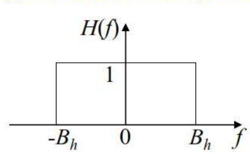
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

The normal laws of _commutativity_, _associativity_, and _distributivity_, apply – i.e.

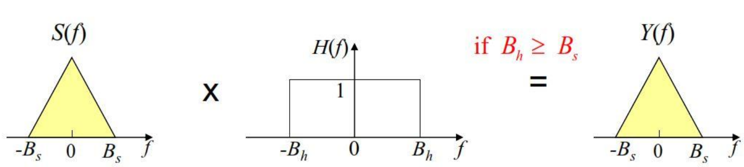


IDEAL LOWPASS SYSTEM

Transfer Function $H(f)$ of an ideal lowpass system:

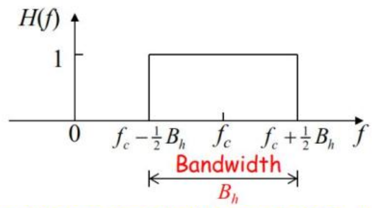


- For a baseband input signal with bandwidth B_s:

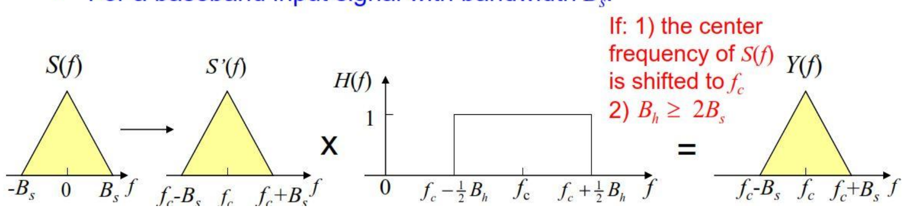


IDEAL BANDPASS SYSTEM

Transfer Function $H(f)$ of an ideal bandpass system:

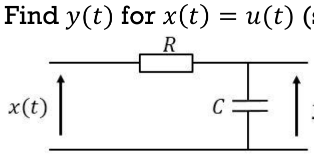


- For a baseband input signal with bandwidth B_s:

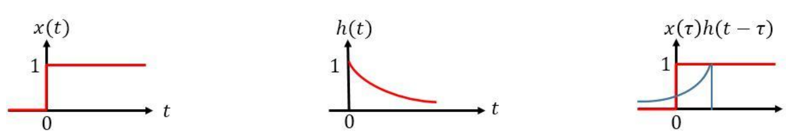


CONVOLUTION EXAMPLE

- Find $y(t)$ for $x(t) = u(t)$ (step input): $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & else \end{cases}$



$\begin{cases} R = 1\Omega \\ C = 1F \end{cases} \Longrightarrow h(t) = \frac{1}{RC}\, e^{\frac{-t}{RC}} u(t) = e^{-t} u(t)$



- $t < 0$: no overlap, hence, $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)\, d\tau = 0$
- $t \geq 0$: overlap, hence, $y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\, d\tau = \int_{0}^{t} (1)e^{-(t-\tau)}\, d\tau = e^{-t}[e^{\tau}]_0^t = 1 - e^{-t}$

CONVOLUTION EXAMPLE CONT…

- Putting (1) and (2) together we get: $y(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & t \geq 0 \end{cases}$