CHAPTER THREE

Transfer Functions

Definition of A Continuous System Transfer Function

This is true for both continuous and discrete systems. We consider continuous transfer functions first, and for single-input, single-output systems only.
The ***transfer function P (s)*** of a continuous system is defined as that factor in the equation for ***Y (s)*** multiplying the transform of the input ***U (s).***

$$Y(s) = P(s)\,U(s)$$

Properties of A Continuous System Transfer Function

The transfer function of a continuous system has several useful properties:

1. It is the Laplace transform of its impulse response $y_\delta(t)\ ,\ t \geq 0$. That is, if the input to a system with transfer function ***P (s)*** is an impulse and all initial values are zero the transform of the output is ***P (s)***.
2. The system transfer function can be determined from the system differential equation by taking the Laplace transform and ignoring all terms arising from initial values. The transfer function ***P (s)*** is then given by

$$P(s) = \frac{Y(s)}{U(s)}$$

3. The system differential equation can be obtained from the transfer function by replacing the s variable with the differential operator ***D*** defined by $D = d/dt$.
4. The stability of a time-invariant linear system can be determined from the characteristic equation. The denominator of the system transfer function is the characteristic polynomial. Consequently, for continuous systems, if all the roots of the denominator have negative real parts, the system is stable.
5. The roots of the denominator are the system poles and the roots of the numerator are the system zeros. The system transfer function can then be specified to within a constant by specifying the system poles and zeros. This constant, usually denoted by K, is the system ***gain factor***.
6. If the system transfer function has no poles or zeros with positive real parts, the system is a ***minimum phase*** system.

Examples

1.

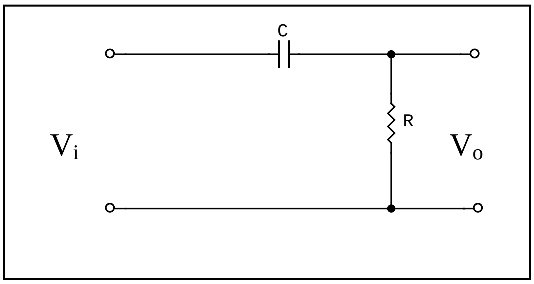
$v_i = iR + \frac{1}{C}\int idt$

$v_o = iR$

$V_i = IR + \frac{1}{sC}I \qquad I = V_i/(R + \frac{1}{sC})$

$V_o = IR \qquad I = \frac{V_o}{R}$

$\frac{V_o}{V_i} = \frac{sCR}{sCR+1}$



2.

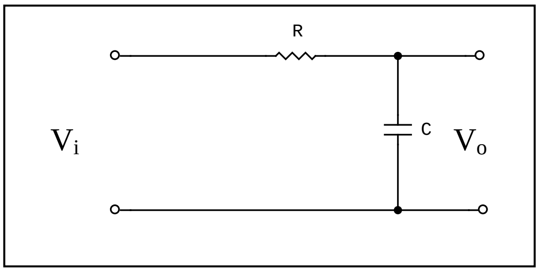
$v_i = iR + \frac{1}{C}\int idt$

$v_o = \frac{1}{C}\int idt$

$V_i = IR + \frac{1}{sC}I \qquad I = V_i/(R + \frac{1}{sC})$

$V_o = \frac{1}{sC}I \qquad I = sCV_o$

$\frac{V_o}{V_i} = \frac{1}{sCR+1}$



3.

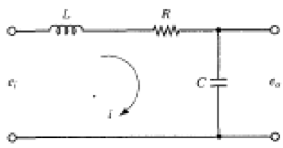
$v_i = L\frac{di}{dt} + iR + \frac{1}{C}\int idt$

$v_o = \frac{1}{C}\int idt$

$V_i = sLI + IR + \frac{1}{sC}I \qquad I = V_i/(sL + R + \frac{1}{sC})$

$V_o = \frac{1}{sC}I \qquad I = sCV_o$

$\frac{V_o}{V_i} = \frac{1}{sC(sL+R+\frac{1}{sC})} \qquad \frac{V_o}{V_i} = \frac{1}{s^2CL+sCR+1}$



4.

$v_i = i_1R_1 + \frac{1}{C_1}\int i_1 - i_2dt$

$0 = i_2R_2 + \frac{1}{C_2}\int i_2dt + \frac{1}{C_1}\int i_2 - i_1dt$

$v_o = \frac{1}{C_2}\int i_2dt$

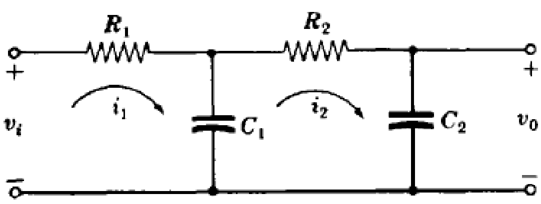
$V_i = I_1\left(R_1 + \frac{1}{sC_1}\right) - \frac{1}{sC_1}I_2$

$0 = I_2\left(R_2 + \frac{1}{sC_1} + \frac{1}{sC_2}\right) - \frac{1}{sC_1}I_1 \qquad I_1 = I_2\left(sC_1R_2 + \frac{C_1}{C_2} + 1\right)$

$V_o = \frac{1}{sC_2}I_2 \qquad I_2 = sC_2V_o$

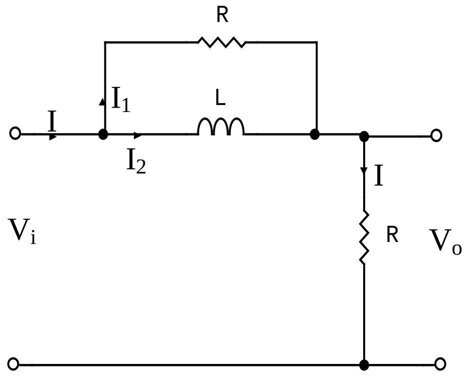
$V_i = sC_2V_o\left(sC_1R_2 + \frac{C_1}{C_2} + 1\right)\left(R_1 + \frac{1}{sC_1}\right) - \frac{1}{sC_1}sC_2V_o$

$V_i = V_o\left[sC_2\left(sC_1R_2 + \frac{C_1}{C_2} + 1\right)\left(R_1 + \frac{1}{sC_1}\right) - \frac{C_2}{C_1}\right]$



$$V_i = V_o\left[sC_2\left(sC_1R_1R_2 + \frac{R_1C_1}{C_2} + R_1 + R_2 + \frac{1}{sC_2} + \frac{1}{sC_1}\right) - \frac{C_2}{C_1}\right]$$
$$V_i = V_o\left[s^2C_1C_2R_2R_1 + sC_1R_1 + sC_2R_1 + sC_2R_2 + 1 + \frac{C_2}{C_1} - \frac{C_2}{C_1}\right]$$
$$V_i = V_o[s^2C_1C_2R_2R_1 + sC_1R_1 + sC_2R_1 + sC_2R_2 + 1]$$
$$\frac{V_o}{V_i} = \left[\frac{1}{s^2C_1C_2R_2R_1 + sC_1R_1 + sC_2R_1 + sC_2R_2 + 1}\right]$$

5.



$i = i_1 + i_2$, $\quad i_1 = \frac{v_i - v_o}{R}$, $\quad i_2 = \frac{v_i - v_o}{L}$, $\quad i = \frac{v_o}{R}$.

$I = I_1 + I_2$, $\quad I_1 = \frac{V_i - V_o}{R}$, $\quad I_2 = \frac{V_i - V_o}{sL}$, $\quad I = \frac{V_o}{R}$.

$\frac{V_o}{R} = \frac{V_i - V_o}{R} + \frac{V_i - V_o}{sL}$, $\quad \frac{V_o}{R} = \frac{V_i}{R} - \frac{V_o}{R} + \frac{V_i}{sL} - \frac{V_o}{sL}$

$\frac{V_o}{R} + \frac{V_o}{R} + \frac{V_o}{sL} = \frac{V_i}{R} + \frac{V_i}{sL}$, $\quad \frac{V_o(2sL+R)}{sLR} = \frac{V_i(sL+R)}{sLR}$

$$\frac{V_o}{V_i} = \frac{(sL+R)}{(2sL+R)}$$