Complex numbers

Cartesian complex numbers

There are several applications of complex numbers in science and engineering, in particular in electrical alternating current theory and in mechanical vector analysis.

There are two main forms of complex number –**Cartesian form and polar form** .

If we can add, subtract, multiply and divide complex numbers in both forms and represent the numbers on an Argand diagram then a.c. theory and vector analysis become considerably easier.

(i) If the quadratic equation *x*2 +2*x* +5 =0 is solved using the quadratic formula then,

$$x = \frac{-2 \pm \sqrt{[(2)^2 - (4)(1)(5)]}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{[-16]}}{2} = \frac{-2 \pm \sqrt{[(16)(-1)]}}{2}$$

$$= \frac{-2 \pm \sqrt{16}\sqrt{-1}}{2} = \frac{-2 \pm 4\sqrt{-1}}{2}$$

$$= -1 \pm 2\sqrt{-1}$$

It is not possible to evaluate $\sqrt{-1}$ in real terms. However, if an operator j is defined as $j=\sqrt{-1}$ then the solution may be expressed as $x=-1\pm j2$.

(ii) $-1+j2$ and $-1-j2$ are known as **complex numbers**. Both solutions are of the form $a+jb$, 'a' being termed the **real part** and jb the **imaginary part**. A complex number of the form $a+jb$ is called **Cartesian complex number**.

(iii) In pure mathematics the symbol i is used to indicate $\sqrt{-1}$ (i being the first letter of the word imaginary). However i is the symbol of electric current in engineering, and to avoid possible confusion the next letter in the alphabet, j, is used to represent $\sqrt{-1}$.

Problem 1. Solve the quadratic equation $x^2+4=0$.

Since $x^2+4=0$ then $x^2=-4$ and $x=\sqrt{-4}$.

i.e., $x=\sqrt{[(-1)(4)]}=\sqrt{(-1)}\sqrt{4}=j(\pm 2)$

$=\pm\, \boldsymbol{j2}$, (since $j=\sqrt{-1}$)

(Note that $\pm j2$ may also be written $\boldsymbol{\pm 2j}$).

Problem 2. Solve the quadratic equation $2x^2+3x+5=0$.

Using the quadratic formula,

$$x=\frac{-3\pm\sqrt{[(3)^2-4(2)(5)]}}{2(2)}$$

$$=\frac{-3\pm\sqrt{-31}}{4}=\frac{-3\pm\sqrt{(-1)}\sqrt{31}}{4}$$

$$=\frac{-3\pm j\sqrt{31}}{4}$$

Hence $\boldsymbol{x=-\frac{3}{4}\pm j\frac{\sqrt{31}}{4}}$ or $\boldsymbol{-0.750\pm j1.392}$,

correct to 3 decimal places.

Problem 3. Evaluate

(a) j^3 (b) j^4 (c) j^{23} (d) $\dfrac{-4}{j^9}$

(a) $j^3=j^2\times j=(-1)\times j=\boldsymbol{-j}$, since $j^2=-1$

(b) $j^4=j^2\times j^2=(-1)\times(-1)=\mathbf{1}$

(c) $j^{23}=j\times j^{22}=j\times(j^2)^{11}=j\times(-1)^{11}$

$=j\times(-1)=\boldsymbol{-j}$

(d) $j^9=j\times j^8=j\times(j^2)^4=j\times(-1)^4$

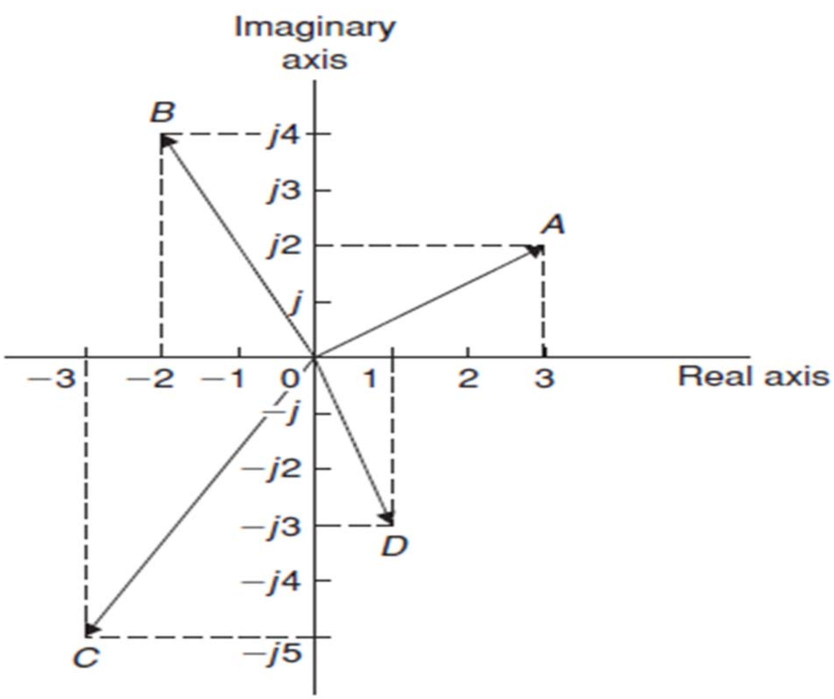
$=j\times 1=\boldsymbol{j}$

$$\text{Hence } \frac{-4}{j^9}=\frac{-4}{j}=\frac{-4}{j}\times\frac{-j}{-j}=\frac{4j}{-j^2}$$

$$=\frac{4j}{-(-1)}=\boldsymbol{4j} \text{ or } \boldsymbol{j4}$$

20.2 The Argand diagram

A complex number may be represented pictorially on rectangular or cartesian axes. The horizontal (or x) axis is used to represent the real axis and the vertical (or y) axis is used to represent the imaginary axis. Such a diagram is called an **Argand diagram**. In Fig. 20.1, the point A represents the complex number $(3+j2)$ and is obtained by plotting the co-ordinates $(3, j2)$ as in graphical work. Figure 20.1 also shows the Argand points B, C and D representing the complex numbers $(-2+j4)$, $(-3-j5)$ and $(1-j3)$ respectively.



20.3 Addition and subtraction of complex numbers

Two complex numbers are added/subtracted by adding/subtracting separately the two real parts and the two imaginary parts.

For example, if $Z_1 = a + jb$ and $Z_2 = c + jd$,

then $\quad Z_1 + Z_2 = (a + jb) + (c + jd)$
$$= (a + c) + j(b + d)$$

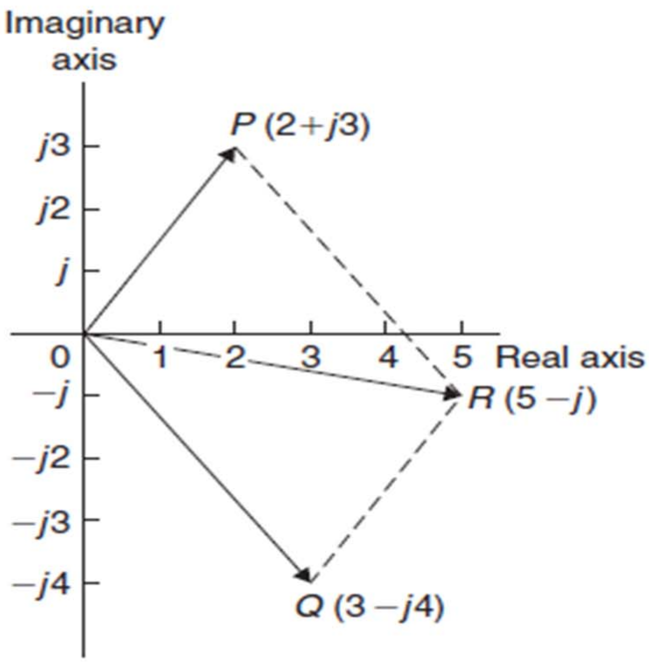
and $\quad Z_1 - Z_2 = (a + jb) - (c + jd)$
$$= (a - c) + j(b - d)$$

Thus, for example,

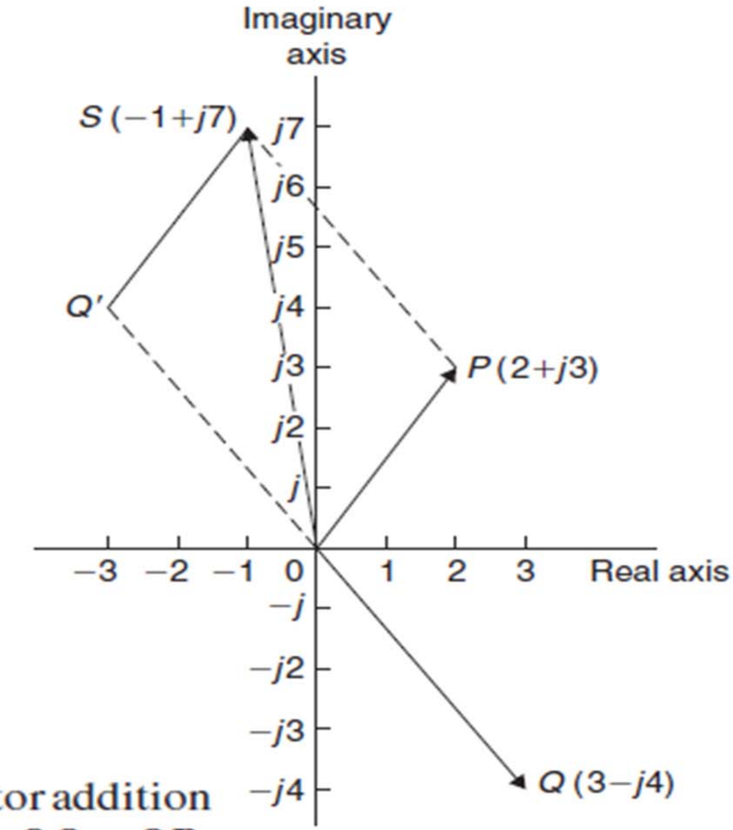
$$(2 + j3) + (3 - j4) = 2 + j3 + 3 - j4$$
$$= \mathbf{5 - j1}$$

and $\quad (2 + j3) - (3 - j4) = 2 + j3 - 3 + j4$
$$= \mathbf{-1 + j7}$$

The addition and subtraction of complex numbers may be achieved graphically as shown in the Argand diagram of Fig. 20.2. $(2 + j3)$ is represented by vector $\boldsymbol{OP}$ and



(a)



(b)

$(3-j4)$ by vector ***OQ***. In Fig. 20.2(a) by vector addition (i.e. the diagonal of the parallelogram) $\boldsymbol{OP}+\boldsymbol{OQ}=\boldsymbol{OR}$. R is the point $(5, -j1)$.

Hence $(2+j3)+(3-j4)=\mathbf{5-j1}$.

In Fig. 20.2(b), vector ***OQ*** is reversed (shown as OQ') since it is being subtracted. (Note $\boldsymbol{OQ}=3-j4$ and $\boldsymbol{OQ'}=-(3-j4)=-3+j4$).
$\boldsymbol{OP}-\boldsymbol{OQ}=\boldsymbol{OP}+\boldsymbol{OQ'}=\boldsymbol{OS}$ is found to be the Argand point $(-1, j7)$.

Hence $(2+j3)-(3-j4)=\mathbf{-1+j7}$

Problem 4. Given $Z_1 = 2 + j4$ and $Z_2 = 3 - j$ determine (a) $Z_1 + Z_2$, (b) $Z_1 - Z_2$, (c) $Z_2 - Z_1$ and show the results on an Argand diagram.

(a) $Z_1 + Z_2 = (2 + j4) + (3 - j)$

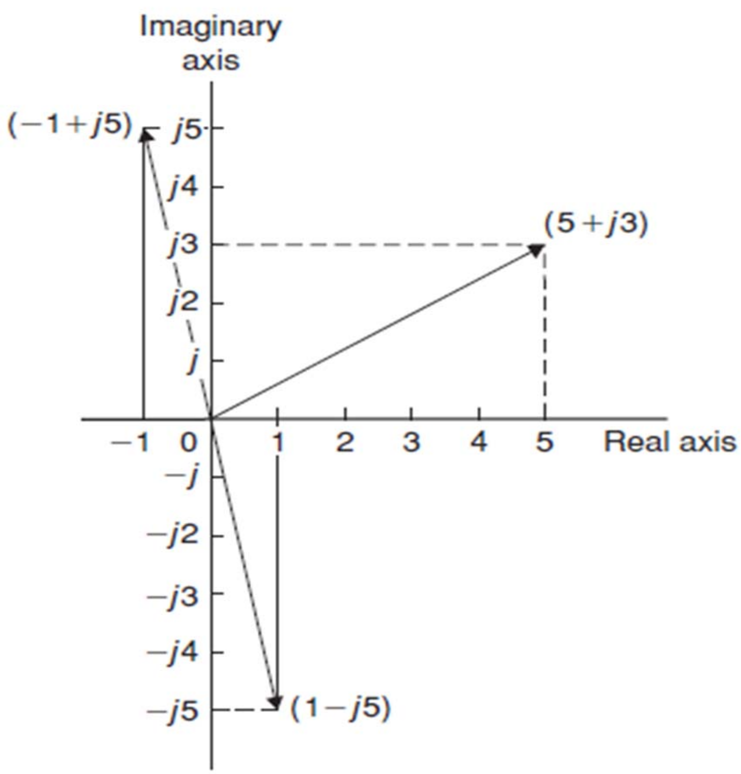
$$= (2 + 3) + j(4 - 1) = \mathbf{5 + j3}$$

(b) $Z_1 - Z_2 = (2 + j4) - (3 - j)$

$$= (2 - 3) + j(4 - (-1)) = \mathbf{-1 + j5}$$

(c) $Z_2 - Z_1 = (3 - j) - (2 + j4)$

$$= (3 - 2) + j(-1 - 4) = \mathbf{1 - j5}$$



20.4 Multiplication and division of complex numbers

(i) **Multiplication of complex numbers** is achieved by assuming all quantities involved are real and then using $j^2 = -1$ to simplify.

Hence $(a + jb)(c + jd)$

$$= ac + a(jd) + (jb)c + (jb)(jd)$$
$$= ac + jad + jbc + j^2bd$$
$$= (ac - bd) + j(ad + bc),$$
$$\text{since } j^2 = -1$$

Thus $(3 + j2)(4 - j5)$

$$= 12 - j15 + j8 - j^2 10$$
$$= (12 - (-10)) + j(-15 + 8)$$
$$= \mathbf{22 - j7}$$

(ii) The **complex conjugate** of a complex number is obtained by changing the sign of the imaginary part. Hence the complex conjugate of $a+jb$ is $a-jb$. The product of a complex number and its complex conjugate is always a real number.

For example,

$$(3+j4)(3-j4)=9-j12+j12-j^2 16$$
$$=9+16=25$$

[$(a+jb)(a-jb)$ may be evaluated 'on sight' as a^2+b^2].

(iii) **Division of complex numbers** is achieved by multiplying both numerator and denominator by the complex conjugate of the denominator.

For example,

$$\frac{2-j5}{3+j4}=\frac{2-j5}{3+j4}\times\frac{(3-j4)}{(3-j4)}$$
$$=\frac{6-j8-j15+j^2 20}{3^2+4^2}$$
$$=\frac{-14-j23}{25}=\mathbf{\frac{-14}{25}-j\frac{23}{25}}$$

or $\mathbf{-0.56-j0.92}$

Problem 5. If $Z_1=1-j3$, $Z_2=-2+j5$ and $Z_3=-3-j4$, determine in $a+jb$ form:

(a) $Z_1 Z_2$ (b) $\frac{Z_1}{Z_3}$

(c) $\frac{Z_1 Z_2}{Z_1+Z_2}$ (d) $Z_1 Z_2 Z_3$

(a) $Z_1 Z_2=(1-j3)(-2+j5)$

$$=-2+j5+j6-j^2 15$$
$$=(-2+15)+j(5+6), \text{ since } j^2=-1,$$
$$=\mathbf{13+j11}$$

(b) $$\frac{Z_1}{Z_3}=\frac{1-j3}{-3-j4}=\frac{1-j3}{-3-j4}\times\frac{-3+j4}{-3+j4}$$
$$=\frac{-3+j4+j9-j^2 12}{3^2+4^2}$$
$$=\frac{9+j13}{25}=\mathbf{\frac{9}{25}+j\frac{13}{25}}$$

or $\mathbf{0.36+j0.52}$

(c) $$\frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(1 - j3)(-2 + j5)}{(1 - j3) + (-2 + j5)}$$

$$= \frac{13 + j11}{-1 + j2}, \text{ from part (a),}$$

$$= \frac{13 + j11}{-1 + j2} \times \frac{-1 - j2}{-1 - j2}$$

$$= \frac{-13 - j26 - j11 - j^2 22}{1^2 + 2^2}$$

$$= \frac{9 - j37}{5} = \mathbf{\frac{9}{5} - j\frac{37}{5}} \text{ or } \mathbf{1.8 - j7.4}$$

(d) $$Z_1 Z_2 Z_3 = (13 + j11)(-3 - j4), \text{ since}$$

$$Z_1 Z_2 = 13 + j11, \text{ from part (a)}$$

$$= -39 - j52 - j33 - j^2 44$$

$$= (-39 + 44) - j(52 + 33)$$

$$= \mathbf{5 - j85}$$