Problem 7. Solve the complex equations:
(a) $2(x + jy) = 6 - j3$
(b) $(1 + j2)(-2 - j3) = a + jb$

(a) $2(x + jy) = 6 - j3$ hence $2x + j2y = 6 - j3$

Equating the real parts gives:

$2x = 6$, i.e. $\boldsymbol{x = 3}$

Equating the imaginary parts gives:

$2y = -3$, i.e. $\boldsymbol{y = -\frac{3}{2}}$

(b) $(1 + j2)(-2 - j3) = a + jb$
$-2 - j3 - j4 - j^2 6 = a + jb$
Hence $4 - j7 = a + jb$
Equating real and imaginary terms gives:

$\boldsymbol{a = 4}$ and $\boldsymbol{b = -7}$

20.6 The polar form of a complex number

(i) Let a complex number z be $x + jy$ as shown in the Argand diagram of Fig. 20.4. Let distance OZ be r and the angle OZ makes with the positive real axis be θ.

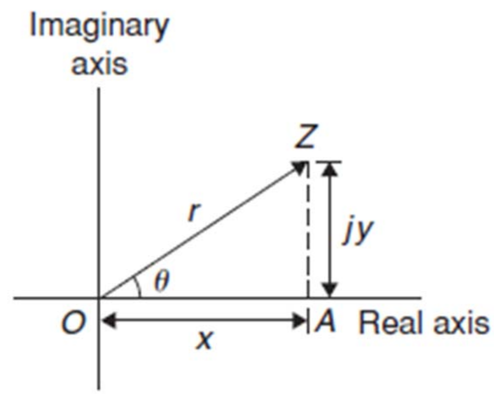
From trigonometry, $x = r\cos\theta$ and

$y = r\sin\theta$

Hence $Z = x + jy = r\cos\theta + jr\sin\theta$

$= r(\cos\theta + j\sin\theta)$

$Z = r(\cos\theta + j\sin\theta)$ is usually abbreviated to $Z = r\angle\theta$ which is known as the **polar form** of a complex number.



(ii) r is called the **modulus** (or magnitude) of Z and is written as mod Z or $|Z|$.
r is determined using Pythagoras' theorem on triangle OAZ in Fig. 20.4,

i.e. $r = \sqrt{(x^2 + y^2)}$

(iii) θ is called the **argument** (or amplitude) of Z and is written as arg Z.

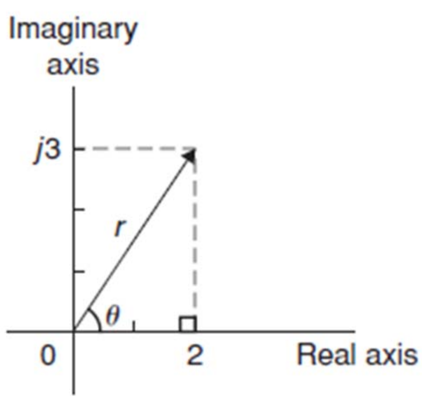
By trigonometry on triangle OAZ,

$$\arg Z = \theta = \tan^{-1}\frac{y}{x}$$

(iv) Whenever changing from cartesian form to polar form, or vice-versa, a sketch is invaluable for determining the quadrant in which the complex number occurs.

Problem 9. Determine the modulus and argument of the complex number $Z = 2 + j3$, and express Z in polar form.

$Z = 2 + j3$ lies in the first quadrant as shown in Fig. 20.5.



Modulus, $|Z| = r = \sqrt{(2^2 + 3^2)} = \sqrt{13}$ or **3.606**, correct to 3 decimal places.

Argument, $\arg Z = \theta = \tan^{-1}\frac{3}{2}$

$= \mathbf{56.31°}$ or $\mathbf{56°19'}$

In polar form, $2 + j3$ is written as $\mathbf{3.606\angle 56.31°}$.

Problem 10. Express the following complex numbers in polar form:

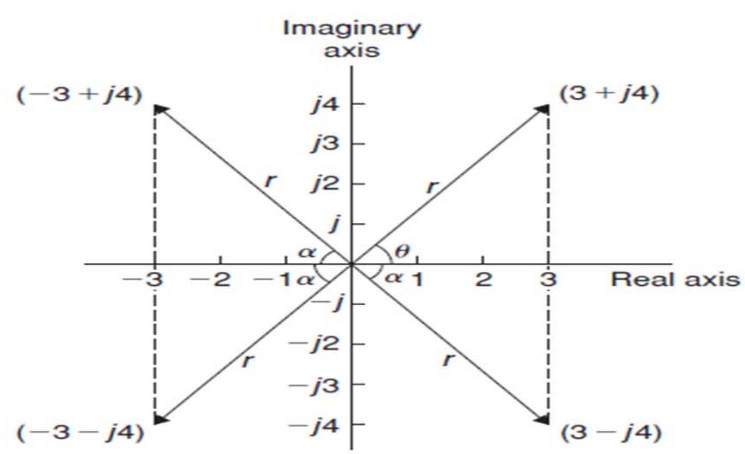
(a) $3 + j4$ (b) $-3 + j4$

(c) $-3 - j4$ (d) $3 - j4$

(a) $3 + j4$ is shown in Fig. 20.6 and lies in the first quadrant.

Modulus, $r = \sqrt{(3^2 + 4^2)} = 5$ and argument $\theta = \tan^{-1}\frac{4}{3} = 53.13°$.

Hence $\mathbf{3 + j4 = 5\angle 53.13°}$



(b) $-3+j4$ is shown in Fig. 20.6 and lies in the second quadrant.

Modulus, $r=5$ and angle $\alpha=53.13°$, from part (a).

Argument $=180°-53.13°=126.87°$ (i.e. the argument must be measured from the positive real axis).

Hence $\mathbf{-3+j4=5\angle 126.87°}$

(c) $-3-j4$ is shown in Fig. 20.6 and lies in the third quadrant.

Modulus, $r=5$ and $\alpha=53.13°$, as above.

Hence the argument $=180°+53.13°=233.13°$, which is the same as $-126.87°$.

Hence $\mathbf{(-3-j4)=5\angle 233.13°}$ **or** $\mathbf{5\angle -126.87°}$

(By convention the **principal value** is normally used, i.e. the numerically least value, such that $-\pi<\theta<\pi$).

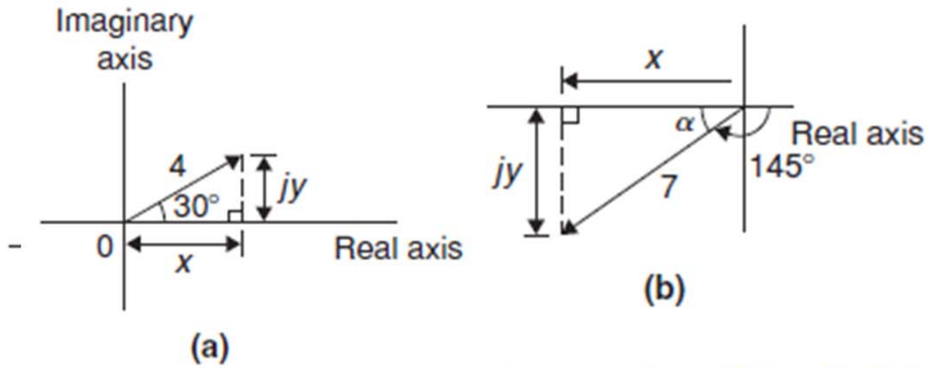
(d) $3-j4$ is shown in Fig. 20.6 and lies in the fourth quadrant.

Modulus, $r=5$ and angle $\alpha=53.13°$, as above.

Hence $\mathbf{(3-j4)=5\angle -53.13°}$

Problem 11. Convert (a) $4\angle 30°$ (b) $7\angle -145°$ into $a+jb$ form, correct to 4 significant figures.

(a) $4\angle 30°$ is shown in Fig. 20.7(a) and lies in the first quadrant.



Using trigonometric ratios, $x=4\cos 30°=3.464$ and $y=4\sin 30°=2.000$.

Hence $\mathbf{4\angle 30°=3.464+j2.000}$

(b) $7\angle 145°$ is shown in Fig. 20.7(b) and lies in the third quadrant.

Angle $\alpha=180°-145°=35°$

Hence $x=7\cos 35°=5.734$

and $y=7\sin 35°=4.015$

Hence $\mathbf{7\angle -145°=-5.734-j4.015}$

Alternatively

$$7\angle -145°=7\cos(-145°)+j7\sin(-145°)$$

$$=\mathbf{-5.734-j4.015}$$

20.7 Multiplication and division in polar form

If $Z_1 = r_1\angle\theta_1$ and $Z_2 = r_2\angle\theta_2$ then:

(i) $Z_1 Z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)$ and

(ii) $\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2}\angle(\theta_1 - \theta_2)$

Problem 12. Determine, in polar form:

(a) $8\angle 25^\circ \times 4\angle 60^\circ$

(b) $3\angle 16^\circ \times 5\angle -44^\circ \times 2\angle 80^\circ$

(a) $8\angle 25^\circ \times 4\angle 60^\circ = (8\times 4)\angle(25^\circ + 60^\circ) = \mathbf{32\angle 85^\circ}$

(b) $3\angle 16^\circ \times 5\angle -44^\circ \times 2\angle 80^\circ$

$= (3\times 5\times 2)\angle[16^\circ + (-44^\circ) + 80^\circ] = \mathbf{30\angle 52^\circ}$

Problem 13. Evaluate in polar form

(a) $\dfrac{16\angle 75^\circ}{2\angle 15^\circ}$ (b) $\dfrac{10\angle\frac{\pi}{4} \times 12\angle\frac{\pi}{2}}{6\angle -\frac{\pi}{3}}$

(a) $\dfrac{16\angle 75^\circ}{2\angle 15^\circ} = \dfrac{16}{2}\angle(75^\circ - 15^\circ) = \mathbf{8\angle 60^\circ}$

(b) $\dfrac{10\angle\frac{\pi}{4} \times 12\angle\frac{\pi}{2}}{6\angle -\frac{\pi}{3}} = \dfrac{10\times 12}{6}\angle\left(\dfrac{\pi}{4} + \dfrac{\pi}{2} - \left(-\dfrac{\pi}{3}\right)\right)$

$= \mathbf{20\angle \dfrac{13\pi}{12}}$ or $\mathbf{20\angle -\dfrac{11\pi}{12}}$ or

$\mathbf{20\angle 195^\circ}$ or $\mathbf{20\angle -165^\circ}$

De Moivre's theorem

21.1 Introduction

From multiplication of complex numbers in polar form,

$$(r\angle\theta) \times (r\angle\theta) = r^2\angle 2\theta$$

Similarly, $(r\angle\theta) \times (r\angle\theta) \times (r\angle\theta) = r^3\angle 3\theta$, and so on. In general, **De Moivre's theorem** states:

$$[r\angle\theta]^n = r^n\angle n\theta$$

The theorem is true for all positive, negative and fractional values of n. The theorem is used to determine powers and roots of complex numbers.

21.2 Powers of complex numbers

For example $[3\angle 20°]^4 = 3^4\angle(4\times 20°) = 81\angle 80°$ by De Moivre's theorem.

Problem 1. Determine, in polar form (a) $[2\angle 35°]^5$ (b) $(-2+j3)^6$.

(a) $[2\angle 35°]^5 = 2^5\angle(5\times 35°)$, from De Moivre's theorem

$= \mathbf{32\angle 175°}$

(b) $(-2+j3) = \sqrt{[(-2)^2+(3)^2]}\angle\tan^{-1}\frac{3}{-2}$

$= \sqrt{13}\angle 123.69°$, since $-2+j3$ lies in the second quadrant

$(-2+j3)^6 = [\sqrt{13}\angle 123.69°]^6$

$= (\sqrt{13})^6\angle(6\times 123.69°)$, by De Moivre's theorem

$= 2197\angle 742.14°$

$= 2197\angle 382.14°$ (since $742.14 \equiv 742.14° - 360° = 382.14°$)

$= \mathbf{2197\angle 22.14°}$ (since $382.14° \equiv 382.14° - 360° = 22.14°$)

or $\mathbf{2197\angle 22°8'}$

Problem 2. Determine the value of $(-7+j5)^4$, expressing the result in polar and rectangular forms.

$$(-7+j5) = \sqrt{[(-7)^2+5^2]}\angle\tan^{-1}\frac{5}{-7}$$

$$= \sqrt{74}\angle 144.46°$$

(Note, by considering the Argand diagram, $-7+j5$ must represent an angle in the second quadrant and **not** in the fourth quadrant.)

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$$(r\angle\theta)\times(r\angle\theta)=r^2\angle 2\theta$$

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$$=\mathbf{32\angle 175°}$$

(b)
$$(-2+j3)=\sqrt{[(-2)^2+(3)^2]}\angle\tan^{-1}\frac{3}{-2}$$

$$=\sqrt{13}\angle 123.69°, \text{ since } -2+j3 \text{ lies in the second quadrant}$$

$$(-2+j3)^6=[\sqrt{13}\angle 123.69°]^6$$

$$=(\sqrt{13})^6\angle(6\times 123.69°), \text{ by De Moivre's theorem}$$

$$=2197\angle 742.14°$$

$$=2197\angle 382.14° (\text{since } 742.14 \equiv 742.14°-360°=382.14°)$$

$$=\mathbf{2197\angle 22.14°} (\text{since } 382.14° \equiv 382.14°-360°=22.14°)$$

$$\text{or } \mathbf{2197\angle 22°8'}$$

Problem 2. Determine the value of $(-7+j5)^4$, expressing the result in polar and rectangular forms.

$$(-7+j5)=\sqrt{[(-7)^2+5^2]}\angle\tan^{-1}\frac{5}{-7}$$

$$=\sqrt{74}\angle 144.46°$$

(Note, by considering the Argand diagram, $-7+j5$ must represent an angle in the second quadrant and **not** in the fourth quadrant.)

Applying De Moivre's theorem:

$$(-7+j5)^4 = [\sqrt{74}\angle 144.46°]^4$$
$$= \sqrt{74^4}\angle 4\times 144.46°$$
$$= 5476\angle 577.84°$$
$$= \mathbf{5476\angle 217.84°}$$

or $\mathbf{5476\angle 217°50'}$ in polar form

Since $r\angle\theta = r\cos\theta + jr\sin\theta$,

$$5476\angle 217.84° = 5476\cos 217.84° + j5476\sin 217.84°$$
$$= -4325 - j3359$$

i.e. $\mathbf{(-7+j5)^4 = -4325 - j3359}$ in rectangular form

21.3 Roots of complex numbers

The **square root** of a complex number is determined by letting $n=1/2$ in De Moivre's theorem,

i.e. $$\sqrt{[r\angle\theta]} = [r\angle\theta]^{\frac{1}{2}} = r^{\frac{1}{2}}\angle\frac{1}{2}\theta = \sqrt{r}\angle\frac{\theta}{2}$$

There are two square roots of a real number, equal in size but opposite in sign.

Problem 3. Determine the two square roots of the complex number $(5+j12)$ in polar and cartesian forms and show the roots on an Argand diagram.

$$(5+j12) = \sqrt{[5^2+12^2]}\angle\tan^{-1}\left(\frac{12}{5}\right)$$
$$= 13\angle 67.38°$$

When determining square roots two solutions result. To obtain the second solution one way is to express $13\angle 67.38°$ also as $13\angle(67.38°+360°)$, i.e. $13\angle 427.38°$. When the angle is divided by 2 an angle less than 360° is obtained.
Hence

$$\sqrt{(5+j12)} = \sqrt{[13\angle 67.38°]} \text{ and } \sqrt{[13\angle 427.38°]}$$
$$= [13\angle 67.38°]^{\frac{1}{2}} \text{ and } [13\angle 427.38°]^{\frac{1}{2}}$$
$$= 13^{\frac{1}{2}}\angle\left(\frac{1}{2}\times 67.38°\right) \text{ and}$$
$$13^{\frac{1}{2}}\angle\left(\frac{1}{2}\times 427.38°\right)$$
$$= \sqrt{13}\angle 33.69° \text{ and } \sqrt{13}\angle 213.69°$$
$$= 3.61\angle 33.69° \text{ and } 3.61\angle 213.69°$$

Thus, in polar form, the two roots are $3.61\angle 33.69°$ and $3.61\angle -146.31°$.

$$\sqrt{13}\angle 33.69° = \sqrt{13}(\cos 33.69° + j\sin 33.69°)$$

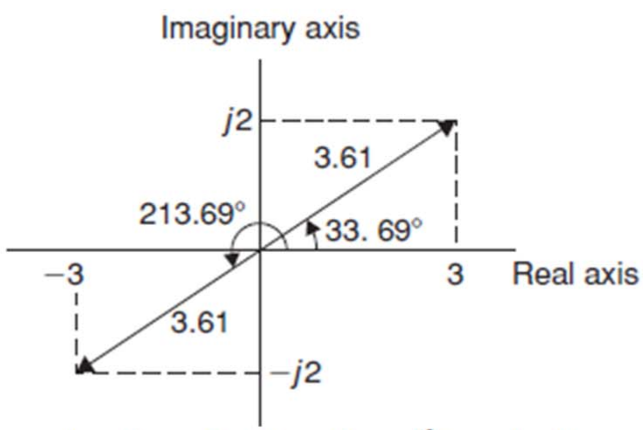
$$= 3.0 + j2.0$$

$$\sqrt{13}\angle 213.69° = \sqrt{13}(\cos 213.69° + j\sin 213.69°)$$

$$= -3.0 - j2.0$$

Thus, in cartesian form the two roots are $\pm(3.0 + j2.0)$.

From the Argand diagram shown in Fig. 21.1 the two roots are seen to be 180° apart, which is always true when finding square roots of complex numbers.



In general, **when finding the n^{th} root of a complex number, there are n solutions**. For example, there are three solutions to a cube root, five solutions to a fifth root, and so on. In the solutions to the roots of a complex number, the modulus, r, is always the same, but the arguments, θ, are different. It is shown in Problem 3 that arguments are symmetrically spaced on an Argand diagram and are $(360/n)°$ apart, where n is the number of the roots required. Thus if one of the solutions to the cube root of a complex number is, say, $5\angle 20°$, the other two roots are symmetrically spaced $(360/3)°$, i.e. 120° from this root and the three roots are $5\angle 20°$, $5\angle 140°$ and $5\angle 260°$.