ELECTROMAGNETIC FIELDS

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Reference : Engineering electromagnetic field : by hayt

Chapter one

Vector Analysis

1.1 Introduction

Vector analysis is a mathematical subject which is much better taught by mathematicians than by engineers. Most junior and senior engineering students, however, have not had the time to take a course in vector analysis, although it is likely that many elementary vector concepts and operations were introduced in the calculus sequence.

1.2 vector notation

In order to distinguish *vectors* (quantities having magnitude and direction) from *scalars* (quantities having magnitude only) the vectors are denoted by boldface symbols. A *unit vector*, one of absolute value (or magnitude or length) 1, will in this book always be indicated by a boldface, lowercase **a**. The unit vector in the direction of a vector **A** is determined by dividing **A** by its absolute value:

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} \quad \text{or} \quad \frac{\mathbf{A}}{A}$$

By use of the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ along the x, y, and z axes of a cartesian coordinate system, an arbitrary vector can be written in *component form*:

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

In terms of components, the absolute value of a vector is defined by

$$|\mathbf{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

1.3 Vector Algebra

1. **Vectors may be added and subtracted.**

$$\mathbf{A} \pm \mathbf{B} = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \pm (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z)$$
$$= (A_x \pm B_x)\mathbf{a}_x + (A_y \pm B_y)\mathbf{a}_y + (A_z \pm B_z)\mathbf{a}_z$$

2. **The associative, distributive, and commutative laws apply.**

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$
$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B} \qquad (k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}$$
$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

3. **The *dot product* of two vectors is, by definition,**

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \qquad \text{(read "A dot B")}$$

where θ is the smaller angle between $\mathbf{A}$ and $\mathbf{B}$. In Example 1 it is shown that

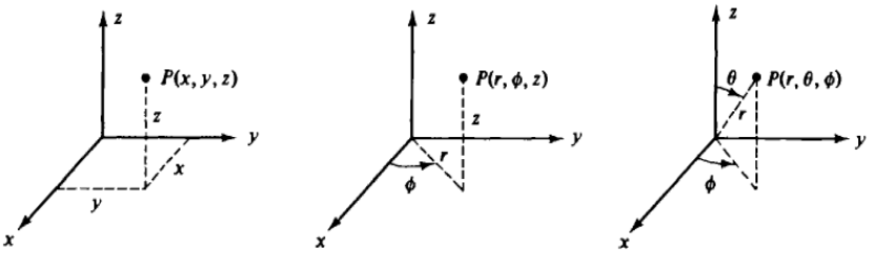
$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

which gives, in particular, $|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$.

1.4 Coordinate system

A problem which has cylindrical or spherical symmetry could be expressed and solved in the familiar cartesian coordinate system. However, the solution would fail to show the symmetry and in most cases would be needlessly complex. Therefore, throughout this book, in addition to the cartesian coordinate system, the circular cylindrical and the spherical coordinate systems will be used. All three will be examined together in order to illustrate the similarities and the differences.

A point P is described by three coordinates, in cartesian (x, y, z), in circular cylindrical (r, ϕ, z), and in spherical (r, θ, ϕ), as shown in Fig. 1-2. The order of specifying the coordinates is important and should be carefully followed. The angle ϕ is the same angle in both the cylindrical and spherical systems. But, in the order of the coordinates, ϕ appears in the second position in cylindrical, (r, ϕ, z), and the third position in spherical, (r, θ, ϕ). The same symbol, r, is used in both cylindrical and spherical for two quite different things. In cylindrical coordinates r measures the distance from the z axis in a plane normal to the z axis, while in the spherical system r measures the distance from the origin to the point. It should be clear from the context of the problem which r is intended.



(*a*) Cartesian (*b*) Cylindrical (*c*) Spherical

Fig. 1-2

The component forms of a vector in the three systems are

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \qquad \text{(cartesian)}$$

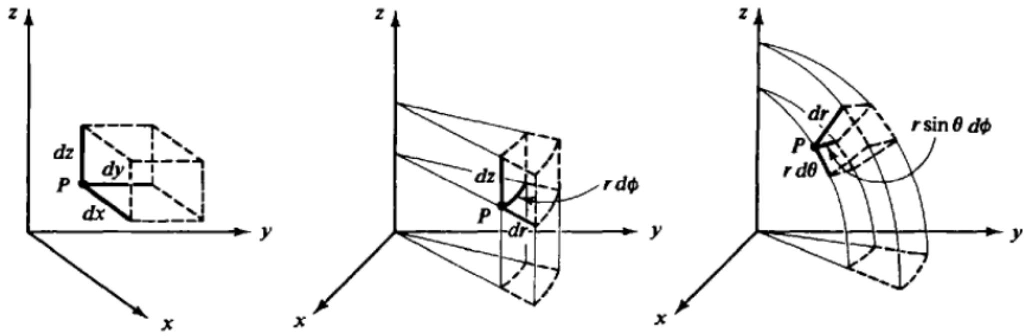
$$\mathbf{A} = A_r\mathbf{a}_r + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$\mathbf{A} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi \qquad \text{(spherical)}$$

1.5 Differential Volume, Surface, And Line elements

There are relatively few problems in electromagnetics that can be solved without some sort of integration—along a curve, over a surface, or throughout a volume. Hence the corresponding differential elements must be clearly understood.

When the coordinates of point P are expanded to $(x + dx, y + dy, z + dz)$ or $(r + dr, \phi + d\phi, z + dz)$ or $(r + dr, \theta + d\theta, \phi + d\phi)$, a differential volume dv is formed. To the first order in infinitesimal quantities the differential volume is, in all three coordinate systems, a rectangular box.



$dv = dx\,dy\,dz$ — (a) Cartesian

$dv = r\,dr\,d\phi\,dz$ — (b) Cylindrical

$dv = r^2 \sin\theta\,dr\,d\theta\,d\phi$ — (c) Spherical

Fig. 1.3

From The Fig 1.3 may also be read the areas of the surface elements that bound the differential volume. For instance, in spherical coordinates, the differential surface element perpendicular to $\mathbf{a}_r$ is

$$dS = (r\,d\theta)(r \sin\theta\,d\phi) = r^2 \sin\theta\,d\theta\,d\phi$$

The differential line element, $d\ell$ is the diagonal through P. Thus

$$d\ell^2 = dx^2 + dy^2 + dz^2 \qquad \text{(cartesian)}$$

$$d\ell^2 = dr^2 + r^2\,d\phi^2 + dz^2 \qquad \text{(cylindrical)}$$

$$d\ell^2 = dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2 \qquad \text{(spherical)}$$

EXAMPLE 1. The dot product obeys the distributive and scalar multiplication laws

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} \qquad \mathbf{A}\cdot k\mathbf{B}=k(\mathbf{A}\cdot\mathbf{B})$$

This being the case,

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}&=(A_x\mathbf{a}_x+A_y\mathbf{a}_y+A_z\mathbf{a}_z)\cdot(B_x\mathbf{a}_x+B_y\mathbf{a}_y+B_z\mathbf{a}_z)\\&=A_xB_x(\mathbf{a}_x\cdot\mathbf{a}_x)+A_yB_y(\mathbf{a}_y\cdot\mathbf{a}_y)+A_zB_z(\mathbf{a}_z\cdot\mathbf{a}_z)\\&\quad+A_xB_y(\mathbf{a}_x\cdot\mathbf{a}_y)+\cdots+A_zB_y(\mathbf{a}_z\cdot\mathbf{a}_y)\end{aligned}$$

However, $\mathbf{a}_x\cdot\mathbf{a}_x=\mathbf{a}_y\cdot\mathbf{a}_y=\mathbf{a}_z\cdot\mathbf{a}_z=1$ because the $\cos\theta$ in the dot product is unity when the angle is zero. And when $\theta=90°$, $\cos\theta$ is zero; hence all other dot products of the unit vectors are zero. Thus

$$\mathbf{A}\cdot\mathbf{B}=A_xB_x+A_yB_y+A_zB_z$$

Chapter two
Coulomb Forces and Electric Field Intensity

Reference : Engineering electromagnetic field :by hayt

2.1 Coulomb‘s Law

There is a force between two charges which is directly proportional to the charge magnitudes and inversely proportional to the square of the separation distance. This is *Coulomb's law*, which was developed from work with small charged bodies and a delicate torsion balance. In vector form, it is stated thus,

$$\mathbf{F}=\frac{Q_1Q_2}{4\pi\epsilon d^2}\mathbf{a}$$

For media other than free space, $\epsilon=\epsilon_0\epsilon_r$, where ϵ_r is the *relative permittivity* or *dielectric constant*. Free space is to be assumed in all problems and examples, as well as the approximate value for ϵ_0, unless there is a statement to the contrary.

For point charges of like sign the Coulomb force is one of repulsion, while for unlike charges the force is attractive. To incorporate this information rewrite Coulomb's law as follows:

$$\mathbf{F}_1=\frac{Q_1Q_2}{4\pi\epsilon_0R_{21}^2}\mathbf{a}_{21}=\frac{Q_1Q_2}{4\pi\epsilon_0R_{21}^3}\mathbf{R}_{21}$$

EXAMPLE 1. Find the force on charge Q_1, 20 μC, due to charge Q_2, -300 μC, where Q_1 is at (0, 1, 2) m and Q_2 at (2, 0, 0) m.

Because 1 C is a rather large unit, charges are often given in microcoulombs (μC), nanocoulombs (nC), or picocoulombs (pC). (See Appendix for the SI prefix system.) Referring to Fig. 2-1,

$$\mathbf{R}_{21} = -2\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a} \qquad R_{21} = \sqrt{(-2)^2 + 1^2 + 2^2} = 3$$

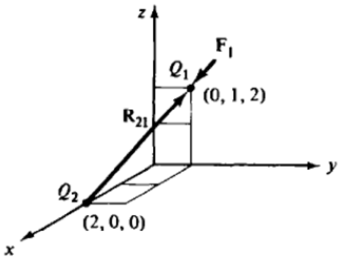


Fig. 2-1

and
$$\mathbf{a}_{21} = \frac{1}{3}(-2\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)$$

Then
$$\mathbf{F}_1 = \frac{(20 \times 10^{-6})(-300 \times 10^{-6})}{4\pi(10^{-9}/36\pi)(3)^2}\left(\frac{-2\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$
$$= 6\left(\frac{2\mathbf{a}_x - \mathbf{a}_y - 2\mathbf{a}_z}{3}\right) \text{ N}$$

2.2 Electric field intensity

For Q at the origin of a spherical coordinate system, the electric field intensity at an arbitrary point P

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

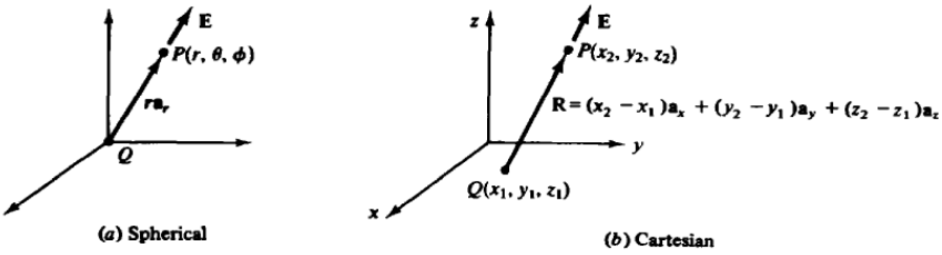


Fig.2.2

2.3 Charge Distributions

Volume Charge

When charge is distributed throughout a specified volume, each charge element contributes to the electric field at an external point. A summation or integration is then required to obtain the total electric field. Even though electric charge in its smallest division is found to be an electron or proton, it is useful to consider continuous (in fact, differentiable) charge distributions and to define a *charge density* by

$$\rho = \frac{dQ}{dv} \qquad (\text{C/m}^3)$$

With reference to volume v in Fig. 2-3, each differential charge dQ produces a differential electric field

$$d\mathbf{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\mathbf{a}_R$$

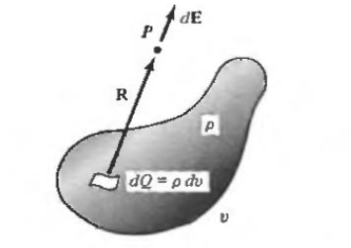


Fig. 2-3

at the observation point P. Assuming that the only charge in the region is contained within the volume, the total electric field at P is obtained by integration over the volume:

$$\mathbf{E} = \int_v \frac{\rho\mathbf{a}_R}{4\pi\epsilon_0 R^2}\,dv$$

Sheet Charge

Charge may also be distributed over a surface or a sheet. Then each differential charge dQ on the sheet results in a differential electric field

$$d\mathbf{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\mathbf{a}_R$$

at point P (see Fig. 2-4). If the *surface charge density* is ρ_s (C/m^2) and if no other charge is present in the region, then the total electric field at P is

$$\mathbf{E} = \int_S \frac{\rho_s\mathbf{a}_R}{4\pi\epsilon_0 R^2}\,dS$$

Line Charge

If charge is distributed over a (curved) line, each differential charge dQ along the line produces a differential electric field

$$d\mathbf{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\mathbf{a}_R$$

at P (See fig 2.4) And if the *line charge density* is ρ_ℓ (C/m), and no other charge is in the region, then the total electric field at P is

$$\mathbf{E} = \int_L \frac{\rho_\ell \mathbf{a}_R}{4\pi\epsilon_0 R^2} d\ell$$

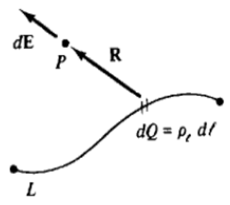


Fig 2.4

2.4 Standard Charge Configurations

Point Charge

As previously determined, the field of a single point charge Q is given by

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r \qquad \text{(spherical coordinates)}$$

Infinite Line Charge

If charge is distributed with *uniform* density ρ_ℓ (C/m) along an *infinite*, *straight* line—which will be chosen as the z axis—then the field is given by

$$\mathbf{E} = \frac{\rho_\ell}{2\pi\epsilon_0 r}\mathbf{a}_r \qquad \text{(cylindrical coordinates)}$$

Example. 2.1

A uniform line charge, infinite in extent, with $\rho_\ell = 20$ nC/m, lies along the z axis. Find $\mathbf{E}$ at (6, 8, 3) m.

In cylindrical coordinates $r = \sqrt{6^2 + 8^2} = 10$ m. The field is constant with z. Thus

$$\mathbf{E} = \frac{20 \times 10^{-9}}{2\pi(10^{-9}/36\pi)(10)}\mathbf{a}_r = 36\mathbf{a}_r \text{ V/m}$$

Infinite Plane Charge

If charge is distributed with *uniform* density ρ_s (C/m^2) over an *infinite plane*, then the field is given by

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_0}\mathbf{a}_n$$

CHAPTER 3

Electric Flux and Gauss Law

Reference : Engineering electromagnetic field :by hayt

3.1 Net Charge in a Region

From

$$dQ = \rho\, dv \quad \text{(C)}$$

$$Q = \int_v \rho\, dv \quad \text{(C)}$$

In general, ρ will not be constant throughout the volume v.

3.2 Electric Flux and Flux Density

Electric flux Ψ, a scalar field, and its density $\mathbf{D}$, a vector field, are useful quantities in solving certain problems, as will be seen in this and subsequent chapters. Unlike $\mathbf{E}$, these fields are not directly measurable; their existence was inferred from nineteenth-century experiments in electrostatics.

By definition, electrix flux Ψ originates on positive charge and terminates on negative charge. In the absence of negative charge, the flux Ψ terminates at infinity. Also by definition, one coulomb of electric charge gives rise to one coulomb of electric flux. Hence

$$\Psi = Q \quad \text{(C)}$$

If in the neighborhood of point P the lines of flux have the direction of the unit vector $\mathbf{a}$ (See fig 3.1)

and if an amount of flux $d\Psi$ crosses the differential area dS, which is a normal to $\mathbf{a}$, then the *electric flux density* at P is

$$\mathbf{D} = \frac{d\Psi}{dS}\mathbf{a} \quad \text{(C/m}^2\text{)}$$

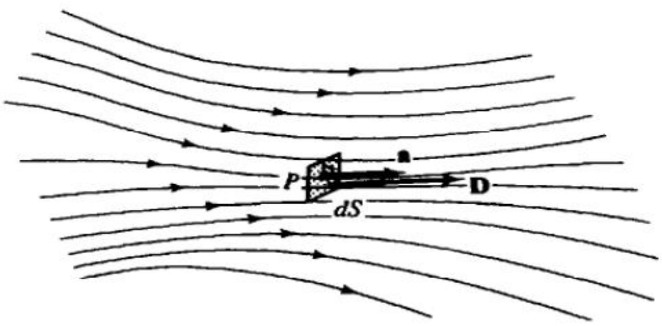


Fig 3.1

density **D** may vary in magnitude and direction from point to point of S; in general, **D** will not be along the normal to S. If, at the surface element dS, **D** makes an angle θ with the normal, then the differential flux crossing dS is given by

$$d\Psi = D\,dS\cos\theta = \mathbf{D}\cdot dS\mathbf{a}_n = \mathbf{D}\cdot d\mathbf{S}$$

3.3 Gauss' Law

Gauss' law states that *The total flux out of a closed surface is equal to the net charge within the surface.* This can be written in integral form as

$$\oint \mathbf{D}\cdot d\mathbf{S} = Q_{enc}$$

3.4 Relation between Flux Density and Electric Field Intensity

Consider a point charge Q (assumed positive, for simplicity) at the origin (See fig 3.2) if this is enclosed by a spherical surface of radius r, then, by symmetry, **D** due to Q is of constant magnitude over the surface and is everywhere normal to the surface. Gauss' law then gives

$$Q = \oint \mathbf{D}\cdot d\mathbf{S} = D\oint dS = D(4\pi r^2)$$

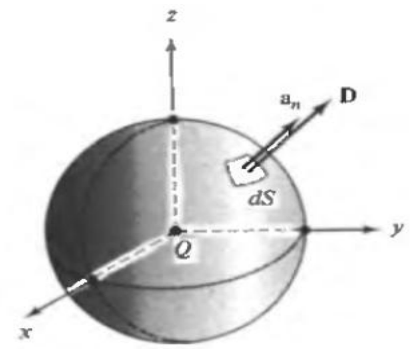


Fig 3.2

from which $D = Q/4\pi r^2$. Therefore

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_n = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

But, from Section 2.2, the electric field intensity due to Q is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

It follows that $\mathbf{D} = \epsilon_0\mathbf{E}$.

More generally, for any electric field in an isotropic medium of permittivity ϵ,

$$\mathbf{D} = \epsilon\mathbf{E}$$

Thus, **D** and **E** fields will have exactly the same form, since they differ only by a factor which is a constant of the medium. While the electric field **E** due to a charge configuration is a function of the permittivity ϵ, the electric fiux density **D** is not. In problems involving multiple dielectrics a distinct advantage will be found in first obtaining **D**, then converting to **E** within each dielectric.

CHAPTER 4

DIVERGENCE and DIVERGENCE THEOEM

Reference : Engineering electromagnetic field :by hayt

4.1 divergence

There are two main indicators of the manner in which a vector field changes from point to point throughout space. The first of these is *divergence*, which will be examined here. It is a scalar and bears a similarity to the derivative of a function.

Divergence of the vector field **A** at the point P is defined by

$$\text{div}\,\mathbf{A} \equiv \lim_{\Delta v \to 0} \frac{\oint \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

Here the integration is over the surface of an infinitesimal volume Δv that shrinks to point P.

4.2 Divergence in Cartesian coordinates

The divergence can be expressed for any vector field in any coordinate system. For the development in cartesian coordinates a cube is selected with edges Δx, Δy, and Δz parallel to the x, y, and z axes, as shown in Fig. 4-1. Then the vector field **A** is defined at P, the corner of the cube with the lowest values of the coordinates x, y, and z.

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

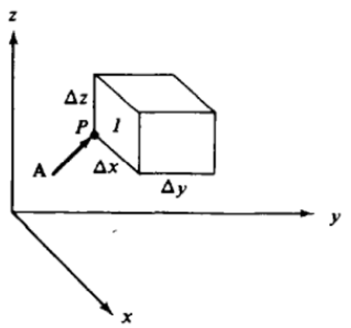


Fig. 4-1

In Fig. 4-2 the cube is turned such that face *1* is in full view; the x components of **A** over the faces to the left and right of *1* are indicated. Since the faces are small,

$$\int_{\text{left face}} \mathbf{A} \cdot d\mathbf{S} \approx -A_x(x)\,\Delta y\,\Delta z$$

$$\int_{\text{right face}} \mathbf{A} \cdot d\mathbf{S} \approx A_x(x+\Delta x)\,\Delta y\,\Delta z$$

$$\approx \left[A_x(x) + \frac{\partial A_x}{\partial x}\Delta x\right]\Delta y\,\Delta z$$

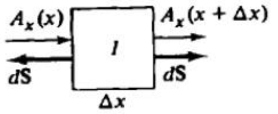


Fig. 4-2

so that the total for these two faces is

$$\frac{\partial A_x}{\partial x}\Delta x\,\Delta y\,\Delta z$$

The same procedure is applied to the remaining two pairs of faces and the results combined.

$$\oint \mathbf{A} \cdot d\mathbf{S} \approx \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

Dividing by $\Delta x\,\Delta y\,\Delta z = \Delta v$ and letting $\Delta v \to 0$, one obtains

$$\text{div}\,\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \text{(cartesian)}$$

The same approach may be used in cylindrical (Problem 4.1) and in spherical coordinates.

$$\text{div}\,\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div}\,\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad \text{(spherical)}$$

Example 4.1

Given the vector field $\mathbf{A} = 5x^2\left(\sin\frac{\pi x}{2}\right)\mathbf{a}_x$, find div **A** at $x = 1$.

$$\text{div}\,\mathbf{A} = \frac{\partial}{\partial x}\left(5x^2 \sin\frac{\pi x}{2}\right) = 5x^2\left(\cos\frac{\pi x}{2}\right)\frac{\pi}{2} + 10x\sin\frac{\pi x}{2} = \frac{5}{2}\pi x^2\cos\frac{\pi x}{2} + 10x\sin\frac{\pi x}{2}$$

and $\text{div}\,\mathbf{A}\big|_{x=1} = 10$.

4.3 Divergence of D

From Gauss' law

$$\frac{\oint \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \frac{Q_{enc}}{\Delta v}$$

In the limit,

$$\lim_{\Delta v \to 0} \frac{\oint \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \text{div}\, \mathbf{D} = \lim_{\Delta v \to 0} \frac{Q_{enc}}{\Delta v} = \rho$$

This important result is one of Maxwell's equations for static fields:

$$\text{div}\, \mathbf{D} = \rho \qquad \text{and} \qquad \text{div}\, \mathbf{E} = \frac{\rho}{\epsilon}$$

if ϵ is constant throughout the region under examination (if not, $\text{div}\, \epsilon\mathbf{E} = \rho$). Thus both **E** and **D** fields will have divergence of zero in any isotropic charge-free region.

4.4 The DEL operator

Vector analysis has its own shorthand, which the reader must note with care. At this point a vector operator, symbolized ∇, is defined *in cartesian coordinates* by

$$\nabla \equiv \frac{\partial(\ )}{\partial x}\mathbf{a}_x + \frac{\partial(\ )}{\partial y}\mathbf{a}_y + \frac{\partial(\ )}{\partial z}\mathbf{a}_z$$

In the calculus a differential operator D is sometimes used to represent d/dx. The symbols $\sqrt{\ }$ and $\int$ are also operators; standing alone, without any indication of what they are to operate on, they look strange. And so ∇, standing alone, simply suggests the taking of certain partial derivatives, each followed by a unit vector. However, when ∇ is dotted with a vector **A**, the result is the divergence of **A**.

$$\nabla \cdot \mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \text{div}\, \mathbf{A}$$

Hereafter, the divergence of a vector field will be written $\nabla \cdot \mathbf{A}$.

4.5 The Divergence Theorem

Gauss' law states that the closed surface integral of $\mathbf{D} \cdot d\mathbf{S}$ is equal to the charge enclosed. If the charge density function ρ is known throughout the volume, then the charge enclosed may be obtained from an integration of ρ throughout the volume. Thus,

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int \rho\, dv = Q_{enc}$$

But $\rho = \nabla \cdot \mathbf{D}$, and so

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{D})\, dv$$

This is the *divergence theorem*, also known as *Gauss' divergence theorem*. It is a three-dimensional analog of Green's theorem for the plane. While it was arrived at from known relationships among **D**, Q, and ρ, the theorem is applicable to any sufficiently regular vector field.

divergence theorem $$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v (\nabla \cdot \mathbf{A})\, dv$$

Of course, the volume v is that which is enclosed by the surface S.

CHAPTER 5

The Electrostatic Field: Work, Energy, and Potential

Reference : Engineering electromagnetic field :by hayt

5.1 WORK DONE IN MOVING A POINT CHARGE

A charge Q experiences a force $\mathbf{F}$ in an electric field $\mathbf{E}$. In order to maintain the charge in equilibrium a force $\mathbf{F}_a$ must be applied in opposition (Fig. 5-1):

$$\mathbf{F} = Q\mathbf{E} \qquad \mathbf{F}_a = -Q\mathbf{E}$$

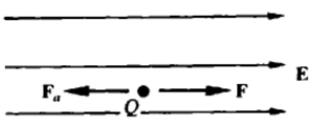


Fig. 5-1

$$dW = \mathbf{F}_a \cdot d\mathbf{l} = -Q\mathbf{E} \cdot d\mathbf{l}$$

$$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \qquad \text{(cartesian)}$$
$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z \qquad \text{(cylindrical)}$$
$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \qquad \text{(spherical)}$$

5.2 CONSERVATIVE PROPERTY OF THE ELECTROSTATIC FIELD

The work done in moving a point charge from one location, B, to another, A, in a static electric field is independent of the path taken. Thus, in terms of Fig. 5-2,

$$\int_{①} \mathbf{E} \cdot d\mathbf{l} = -\int_{②} \mathbf{E} \cdot d\mathbf{l} \qquad \text{or} \qquad \oint_{①-②} \mathbf{E} \cdot d\mathbf{l} = 0$$

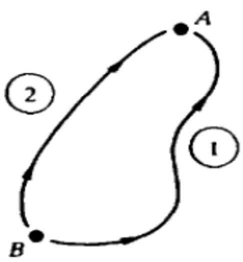


Fig. 5-2

where the last integral is over the *closed contour* formed by ① described positively and ② described negatively. Conversely, if a vector field $\mathbf{F}$ has the property that $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ over *every* closed contour, then the value of any line integral of $\mathbf{F}$ is determined solely by the endpoints of the path. Such a field $\mathbf{F}$ is called *conservative*; it can be shown that a criterion for the conservative property is that the curl of $\mathbf{F}$ vanish identically.

EXAMPLE 1. An electrostatic field is given by $\mathbf{E} = (x/2 + 2y)\mathbf{a}_x + 2x\mathbf{a}_y$ (V/m). Find the work done in moving a point charge $Q = -20\ \mu C$ (*a*) from the origin to (4, 0, 0) m, and (*b*) from (4, 0, 0) m to (4, 2, 0) m.

(*a*) The first path is along the x axis, so that $d\mathbf{l} = dx\ \mathbf{a}_x$.

$$dW = -Q\mathbf{E}\cdot d\mathbf{l} = (20\times 10^{-6})\left(\frac{x}{2} + 2y\right)dx$$

$$W = (20\times 10^{-6})\int_0^4 \left(\frac{x}{2} + 2y\right)dx = 80\ \mu J$$

(*b*) The second path is in the $\mathbf{a}_y$ direction, so that $d\mathbf{l} = dy\mathbf{a}_y$.

$$W = (20\times 10^{-6})\int_0^2 2x\,dy = 320\ \mu J$$

5.3 ELECTRIC POTENTIAL BETWEEN TWO POINTS

The *potential* of point A with respect to point B is defined as the work done in moving a unit positive charge, Q_u, from B to A.

$$V_{AB} = \frac{W}{Q_u} = -\int_B^A \mathbf{E}\cdot d\mathbf{l} \quad \text{(J/C or V)}$$

5.4 POTENTIAL OF A POINT CHARGE

Since the electric field due to a point charge Q is completely in the radial direction,

$$V_{AB} = -\int_B^A \mathbf{E}\cdot d\mathbf{l} = -\int_{r_B}^{r_A} E_r\,dr = -\frac{Q}{4\pi\epsilon_0}\int_{r_B}^{r_A}\frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

5.5 POTENTIAL OF A CHARGE DISTRIBUTION

If charge is distributed throughout some finite volume with a known charge density ρ (C/m^3), then the potential at some external point can be determined. To do so, a differential charge at a general point within the volume is identified

.

$$dV = \frac{dQ}{4\pi\epsilon_0 R}$$

Integration over the volume gives the total potential at P:

$$V = \int_{\text{vol}} \frac{\rho\,dv}{4\pi\epsilon_0 R}$$

5.6 GRADIENT

At this point another operation of vector analysis is introduced. Figure (5.3) shows two neighboring points, M and N, of the region in which a scalar function V is defined. The vector separation of the two points is

$$d\mathbf{r} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

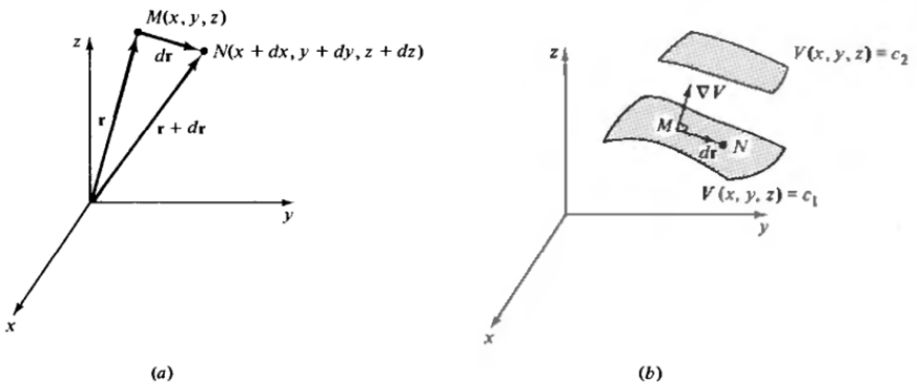


Fig (5.3)

It is noted that each term contains the partial derivative of V with respect to distance in the direction of that particular unit vector.

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cartesian)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{\partial V}{r\,\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{\partial V}{r\,\partial \theta}\mathbf{a}_\theta + \frac{\partial V}{r \sin\theta\,\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}$$

While ∇V is written for grad V in any coordinate system, it must be remembered that the del operator is defined only in cartesian coordinates.