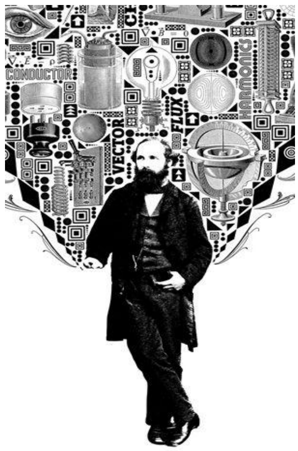

Electricity and Magnetism

An Introduction to Vectors and Calculus

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ELECTROMAGNETIC FIELDS By Ahmed W.

Lecture's Contents Part - II

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   - Vector Line Integral
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1. Line and Surface Integrals

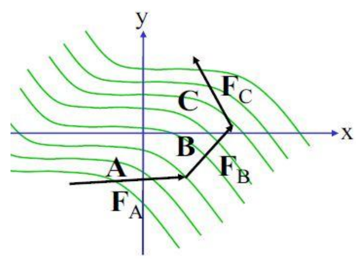
– **Vector Line Integrals**

$$\int_l \vec{\mathbf{F}} \cdot d\hat{\mathbf{l}} = \int_A \vec{\mathbf{F}}_A \cdot d\hat{\mathbf{l}} + \int_B \vec{\mathbf{F}}_B \cdot d\hat{\mathbf{l}} + \int_C \vec{\mathbf{F}}_C \cdot d\hat{\mathbf{l}}$$

$$= \vec{\mathbf{F}}_A \cdot \vec{\mathbf{A}} + \vec{\mathbf{F}}_B \cdot \vec{\mathbf{B}} + \vec{\mathbf{F}}_C \cdot \vec{\mathbf{C}}$$

Vector Field $\vec{\mathbf{F}} = \vec{\mathbf{F}}_A + \vec{\mathbf{F}}_B + \vec{\mathbf{F}}_C$

Vector Line $l = \vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}}$



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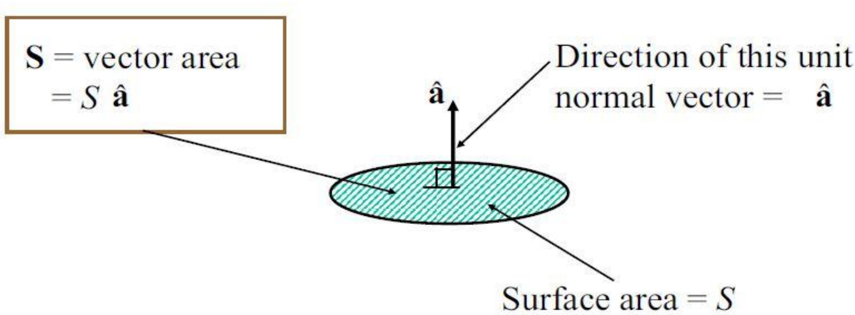
1. Line and Surface Integrals

– **Vector Surface Integrals**

Definition of a vector area:



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1. Basic Vector Algebra

– Vector Surface Integrals

$$\int_S \vec{\mathbf{F}} \cdot d\hat{\mathbf{s}} = \int_{S_A} \vec{\mathbf{F}}_A \cdot d\hat{\mathbf{s}} + \int_{S_B} \vec{\mathbf{F}}_B \cdot d\hat{\mathbf{s}} + \int_{S_C} \vec{\mathbf{F}}_C \cdot d\hat{\mathbf{s}}$$

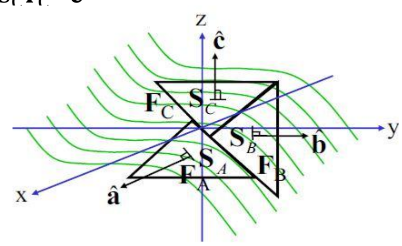
$$= \vec{\mathbf{F}}_A \cdot \overrightarrow{\mathbf{S}_A} + \vec{\mathbf{F}}_B \cdot \overrightarrow{\mathbf{S}_B} + \vec{\mathbf{F}}_C \cdot \overrightarrow{\mathbf{S}_C}$$

$$= S_A \vec{\mathbf{F}}_A \cdot \hat{\mathbf{a}} + S_B \vec{\mathbf{F}}_B \cdot \hat{\mathbf{b}} + S_C \vec{\mathbf{F}}_C \cdot \hat{\mathbf{c}}$$

$\vec{\mathbf{S}}$= Vector Area

$= \overrightarrow{\mathbf{S}_A} + \overrightarrow{\mathbf{S}_B} + \overrightarrow{\mathbf{S}_C}$

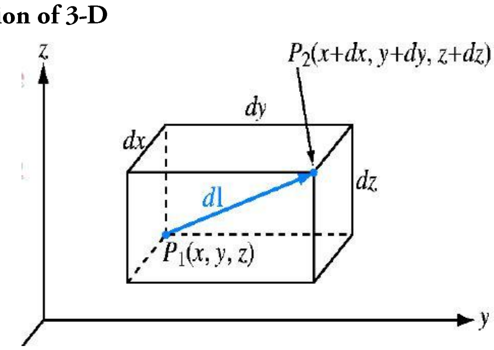
$= S_A\, \hat{\mathbf{a}} + S_B\, \hat{\mathbf{b}} + S_C\, \hat{\mathbf{c}}$



2. Gradient of a Scalar Field

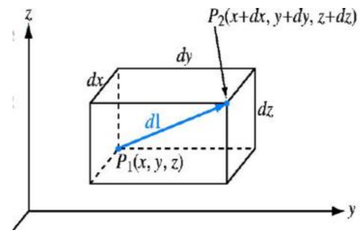
– Concept of Gradient

- **Let us assume a temperature (T) as a function of space parameters.**
- **Gradient is an extension of the derivative** dT/dz
- **However, if the (T) is a function of 3-D position (x, y, z), then we need to use Gradient to describe the increase if the temperature (T) as a function of (x, y, z).**



2. Gradient of a Scalar Field

– **Del or Gradient Operator**



$$d\hat{\mathbf{l}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$= \hat{\mathbf{x}}\frac{\partial T}{\partial x}\cdot d\hat{\mathbf{l}} + \hat{\mathbf{y}}\frac{\partial T}{\partial y}\cdot d\hat{\mathbf{l}} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\cdot d\hat{\mathbf{l}} = \left(\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right)\cdot d\hat{\mathbf{l}}$$

$$= \nabla T\cdot d\hat{\mathbf{l}}$$

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \quad \textbf{or} \quad \boldsymbol{\nabla} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

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2. Gradient of a Scalar Field

– **Gradient Operator of Cylindrical and Spherical Systems**

- **Spherical System**

$$\nabla = \frac{\partial}{\partial R}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial}{\partial \Theta}\hat{\boldsymbol{\Theta}} + \frac{1}{R\sin\theta}\frac{\partial}{\partial \Phi}\hat{\boldsymbol{\Phi}}$$

- **Cylindrical System**

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \varphi}\hat{\boldsymbol{\Phi}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

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3. Divergence Theorem of a Vector Field

– Concept of Divergence

- **The dot product of del operator ∇ and the vector $\vec{A}$ gives:**

$$\text{Div}\,\vec{\mathbf{A}} = \nabla \cdot \vec{\mathbf{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial x}$$

- **In the Cylindrical Coordinate System:**

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- **In the Spherical Coordinate System:**

$$\nabla \cdot \boldsymbol{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

3. Divergence Theorem of a Vector Field

– The Divergence Theorem

- **This powerful theorem converts a surface integral into equivalent volume integral.**
- **Figure (a) is adjoining a number of incremental volumes of square shapes. We made a macroscopic volume V with enclosing surface S.**
- **Any interior shared surface between two incremental volumes has a flux of ZERO (one entering the volume and another leaving it).**
- **Nonzero flux obtained only for those surfaces which bound the outer surface S of V.**
- **All interior flux of volumes cancels each other.**
- **The divergence theorem states:**

$$\Phi = \oint_S \vec{\mathbf{A}} \cdot d\hat{\mathbf{s}} = \int_V \nabla \cdot \vec{\mathbf{A}}\ d\hat{\mathbf{v}}$$

4. Curl and Del Operator

– Curl Operator and Curl Identities

- **The Curl Operator is:**

$$\nabla \times \vec{\mathbf{B}} = \text{curl}\,\vec{\mathbf{B}} = \log_{\Delta s \to \mathbf{0}} \frac{1}{\Delta s} [\hat{\mathbf{n}} \oint \vec{\mathbf{B}} \cdot \mathrm{d}\hat{\mathbf{l}}$$

- **Distributive property of two vectors:**

$$\nabla \times (\vec{\mathbf{A}} + \vec{\mathbf{B}}) = \nabla \times \vec{\mathbf{A}} + \nabla \times \vec{\mathbf{B}}$$

- **Divergence of a curl of a vector field:**

$$\nabla \cdot (\nabla \times \vec{\mathbf{A}}) = 0$$

- **Curl of a gradient of a scalar field**

$$\nabla \times (\nabla \vec{\mathbf{A}}) = 0$$

4. Curl and Del Operator

– Stokes's Theorem

- **The Stokes's theorem converts the integral of the curl of a vector over an open surface (S) into a line integral of the vector along the contour (C) bounding the surface (S)**

$$(\nabla \times \vec{\mathbf{B}}) \cdot \mathrm{d}\hat{\mathbf{s}} = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\hat{\mathbf{l}}$$

- **If the $\nabla \times \vec{\mathbf{B}} = \mathbf{0}$, then the field is said to be CONSERVATIVE.**

5. Laplacian Operator

- Laplacian Operator in Cartesian Coordinates

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} = \hat{\mathbf{x}} \nabla^2 E_x + \hat{\mathbf{y}} \nabla^2 E_y + \hat{\mathbf{z}} \nabla^2 E_z$$

- Laplacian Operator in Cylindrical Coordinates

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

- Laplacian Operator in Spherical Coordinates

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2}$$

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