Mathematics

References

1. Calculus By William E. Boyce and Richard C. Diprima 1988
2. Thomas Calculus by George b. Thomas (Twelfth Edition)

The polar coordinates

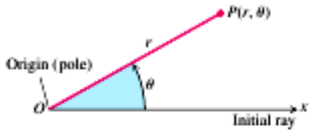


FIGURE 10.35 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. This has interesting consequences for graphing, as we will see in the next section.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (Figure 10.35). Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

Polar Coordinates

$P(r, \theta)$

Directed distance from O to P — Directed angle from initial ray to OP

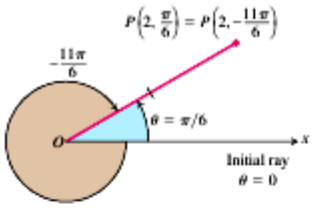


FIGURE 10.36 Polar coordinates are not unique.

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2$, $\theta = \pi/6$. It also has coordinates $r = 2$, $\theta = -11\pi/6$ (Figure 10.36). There are occasions when we wish to allow r to be negative. That is why we use directed distance in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units (Figure 10.37). It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2$, $\theta = \pi/6$.

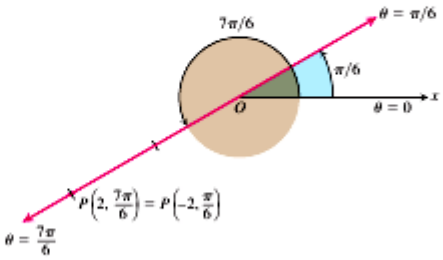


FIGURE 10.37 Polar coordinates can have negative r-values.

EXAMPLE 1 Finding Polar Coordinates

Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 10.38). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

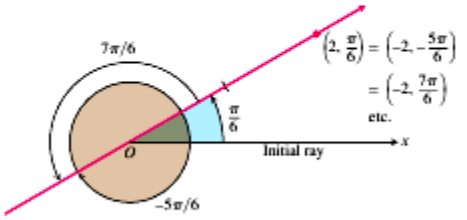


FIGURE 10.38 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \ldots.$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \ldots.$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

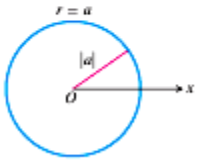


FIGURE 10.39 The polar equation for a circle is $r = a$.

Polar Equations and Graphs

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Figure 10.39).

If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞, the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray.

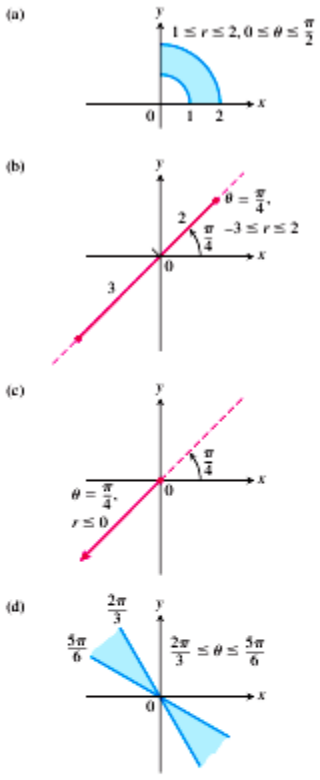


FIGURE 10.40 The graphs of typical inequalities in r and θ (Example 3).

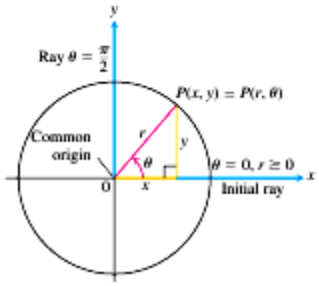


FIGURE 10.41 The usual way to relate polar and Cartesian coordinates.

Equation	Graph
$r = a$	Circle radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

EXAMPLE 2 Finding Polar Equations for Graphs

(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Figure 10.38. ■

Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments, and rays.

EXAMPLE 3 Identifying Graphs

Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2$ and $0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \dfrac{\pi}{4}$

(c) $r \le 0$ and $\theta = \dfrac{\pi}{4}$

(d) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

Solution The graphs are shown in Figure 10.40. ■

Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive y-axis (Figure 10.41). The two coordinate systems are then related by the following equations.

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2$$

The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ. On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each selection, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y). The other polar coordinate representations for the point can be determined from these two, as in Example 1.

EXAMPLE 4 Equivalent Equations

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

With some curves, we are better off with polar coordinates; with others, we aren't. ■

EXAMPLE 5 Converting Cartesian to Polar

Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$ (Figure 10.42).

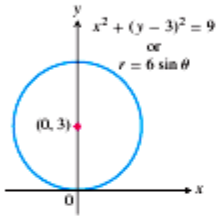


FIGURE 10.42 The circle in Example 5.

Solution

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y-3)^2.$$
$$x^2 + y^2 - 6y = 0 \qquad \text{The 9's cancel.}$$
$$r^2 - 6r\sin\theta = 0 \qquad x^2 + y^2 = r^2$$
$$r = 0 \quad \text{or} \quad r - 6\sin\theta = 0$$
$$r = 6\sin\theta \qquad \text{Includes both possibilities}$$

We will say more about polar equations of conic sections in Section 10.8. ■

EXAMPLE 6 Converting Polar to Cartesian

Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation: $r\cos\theta = -4$

$x = -4$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation: $r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x$

$x^2 - 4x + y^2 = 0$

$x^2 - 4x + 4 + y^2 = 4$ Completing the square

$(x - 2)^2 + y^2 = 4$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation: $r(2\cos\theta - \sin\theta) = 4$

$2r\cos\theta - r\sin\theta = 4$

$2x - y = 4$

$y = 2x - 4$

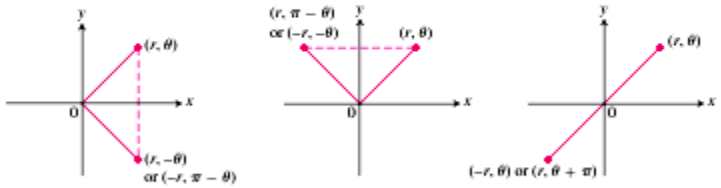
The graph: Line, slope $m = 2$, y-intercept $b = -4$

Graphing in polar coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry

Figure 10.43 illustrates the standard polar coordinate tests for symmetry.



(a) About the x-axis (b) About the y-axis (c) About the origin

FIGURE 10.43 Three tests for symmetry in polar coordinates.

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.43a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.43b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.43c).

Slope

The slope of a polar curve $r = f(\theta)$ is given by dy/dx, not by $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$