# Graphing in polar coordinates

This section describes techniques for graphing equations in polar coordinates.

## Symmetry

Figure 10.43 illustrates the standard polar coordinate tests for symmetry.

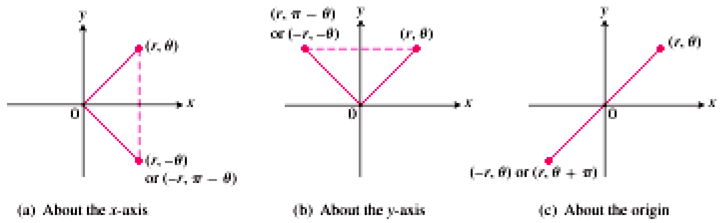



**FIGURE 10.43** Three tests for symmetry in polar coordinates.

> **Symmetry Tests for Polar Graphs**
>
> 1. *Symmetry about the x-axis:* If the point $(r, \theta)$ lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.43a).
> 2. *Symmetry about the y-axis:* If the point $(r, \theta)$ lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.43b).
> 3. *Symmetry about the origin:* If the point $(r, \theta)$ lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.43c).

## Slope

The slope of a polar curve $r = f(\theta)$ is given by $dy/dx$, not by $r' = df/d\theta$. To see why, think of the graph of $f$ as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

**EXAMPLE 2** Graph the Curve $r^2 = 4\cos\theta$.

**Solution** The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running $\theta$ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the $x$-axis because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.}\end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow (-r)^2 = 4\cos\theta \\ &\Rightarrow (-r, \theta) \text{ on the graph.}\end{aligned}$$

Together, these two symmetries imply symmetry about the $y$-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

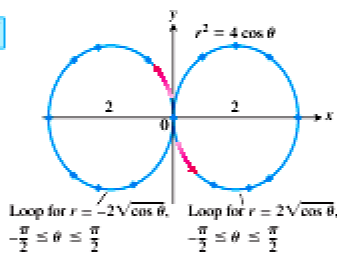
For each value of $\theta$ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of $r$:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve (Figure 10.45). ■

| $\theta$ | $\cos\theta$ | $r = \pm 2\sqrt{\cos\theta}$ |
|---|---|---|
| 0 | 1 | $\pm 2$ |
| $\pm\frac{\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $\approx \pm 1.9$ |
| $\pm\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\approx \pm 1.7$ |
| $\pm\frac{\pi}{3}$ | $\frac{1}{2}$ | $\approx \pm 1.4$ |
| $\pm\frac{\pi}{2}$ | 0 | 0 |

(a)




(b)

**FIGURE 10.45** The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing $\theta$. The values of $r$ in the table are rounded (Example 2).

## A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of $(r, \theta)$-values, plot the corresponding points, and connect them in order of increasing $\theta$. This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. Another method of graphing that is usually quicker and more reliable is to

1. first graph $r = f(\theta)$ in the *Cartesian* $r\theta$-plane,
2. then use the Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph.

This method is better than simple point plotting because the first Cartesian graph, even when hastily drawn, shows at a glance where $r$ is positive, negative, and nonexistent, as well as where $r$ is increasing and decreasing. Here's an example.

**EXAMPLE 3** A Lemniscate

Graph the curve

$$r^2 = \sin 2\theta.$$

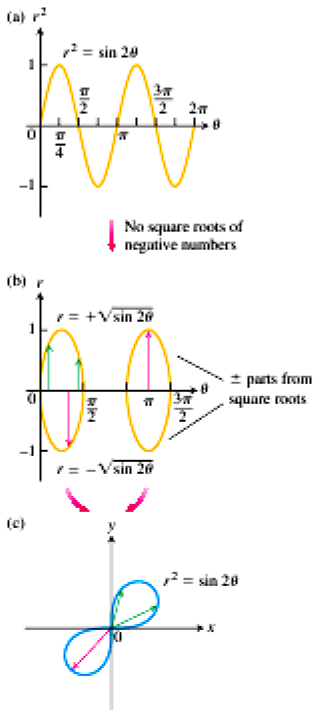



**FIGURE 10.46** To plot $r = f(\theta)$ in the Cartesian $r\theta$-plane in (b), we first plot $r^2 = \sin 2\theta$ in the $r^2\theta$-plane in (a) and then ignore the values of $\theta$ for which $\sin 2\theta$ is negative. The radii from the sketch in (b) cover the polar graph of the lemniscate in (c) twice (Example 3).

**Solution** Here we begin by plotting $r^2$ (not $r$) as a function of $\theta$ in the Cartesian $r^2\theta$-plane. See Figure 10.46a. We pass from there to the graph of $r = \pm\sqrt{\sin 2\theta}$ in the $r\theta$-plane (Figure 10.46b), and then draw the polar graph (Figure 10.46c). The graph in Figure 10.46b "covers" the final polar graph in Figure 10.46c twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves. The double covering does no harm, however, and we actually learn a little more about the behavior of the function this way. ■

## Finding Points Where Polar Graphs Intersect

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve. Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. One sure way to identify all the points of intersection is to graph the equations.

**EXAMPLE 4** Deceptive Polar Coordinates

Show that the point $(2, \pi/2)$ lies on the curve $r = 2\cos 2\theta$.

**Solution** It may seem at first that the point $(2, \pi/2)$ does not lie on the curve because substituting the given coordinates into the equation gives

$$2 = 2\cos 2\left(\frac{\pi}{2}\right) = 2\cos\pi = -2,$$

which is not a true equality. The magnitude is right, but the sign is wrong. This suggests looking for a pair of coordinates for the same given point in which $r$ is negative, for example, $(-2, -(\pi/2))$. If we try these in the equation $r = 2\cos 2\theta$, we find

$$-2 = 2\cos 2\left(-\frac{\pi}{2}\right) = 2(-1) = -2,$$

and the equation is satisfied. The point $(2, \pi/2)$ does lie on the curve. ■

**EXAMPLE 5** Elusive Intersection Points

Find the points of intersection of the curves

$$r^2 = 4\cos\theta \quad \text{and} \quad r = 1 - \cos\theta.$$

**Solution** In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others are found by graphing. (Also, see Exercise 49.)

If we substitute $\cos\theta = r^2/4$ in the equation $r = 1 - \cos\theta$, we get

$$r = 1 - \cos\theta = 1 - \frac{r^2}{4}$$

$$4r = 4 - r^2$$

$$r^2 + 4r - 4 = 0$$

$$r = -2 \pm 2\sqrt{2}. \qquad \text{Quadratic formula}$$

The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to belong to either curve. The values of $\theta$ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\begin{aligned}\theta &= \cos^{-1}(1 - r) && \text{From } r = 1 - \cos\theta\\ &= \cos^{-1}\left(1 - \left(2\sqrt{2} - 2\right)\right) && \text{Set } r = 2\sqrt{2} - 2.\\ &= \cos^{-1}\left(3 - 2\sqrt{2}\right)\\ &= \pm 80°. && \text{Rounded to the nearest degree}\end{aligned}$$

We have thus identified two intersection points: $(r, \theta) = (2\sqrt{2} - 2, \pm 80°)$.

If we graph the equations $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$ together (Figure 10.47), as we can now do by combining the graphs in Figures 10.44 and 10.45, we see that the curves also intersect at the point $(2, \pi)$ and the origin. Why weren't the $r$-values of these points revealed by the simultaneous solution? The answer is that the points (0, 0) and $(2, \pi)$ are not on the curves "simultaneously." They are not reached at the same value of $\theta$. On the curve $r = 1 - \cos\theta$, the point $(2, \pi)$ is reached when $\theta = \pi$. On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$, where it is identified not by the coordinates $(2, \pi)$, which do not satisfy the equation, but by the coordinates $(-2, 0)$, which do. Similarly, the cardioid reaches the origin when $\theta = 0$, but the curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$. ■

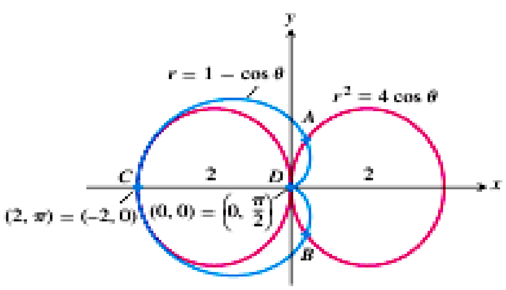



**FIGURE 10.47** The four points of intersection of the curves $r = 1 - \cos\theta$ and $r^2 = 4\cos\theta$ (Example 5). Only $A$ and $B$ were found by simultaneous solution. The other two were disclosed by graphing.

## Area and length in polar coordinates

**Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} (f(\theta))^2 \, d\theta.$$

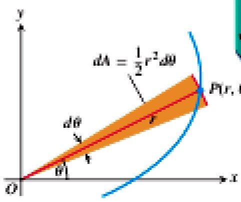



FIGURE 10.49 The area differential $dA$ for the curve $r = f(\theta)$.

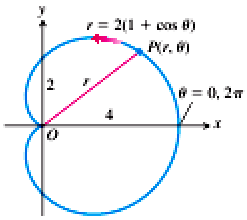



FIGURE 10.50 The cardioid in Example 1.

**EXAMPLE 1** Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

**Solution** We graph the cardioid (Figure 10.50) and determine that the radius $OP$ sweeps out the region exactly once as $\theta$ runs from 0 to $2\pi$. The area is therefore

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta \\
&= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \\
&= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta \\
&= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.
\end{aligned}$$

**EXAMPLE 2** Finding Area

Find the area inside the smaller loop of the limaçon

$$r = 2\cos\theta + 1.$$

**Solution** After sketching the curve (Figure 10.51), we see that the smaller loop is traced out by the point $(r, \theta)$ as $\theta$ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the $x$-axis (the equation is unaltered when we replace $\theta$ by $-\theta$), we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2 \, d\theta = \int_{2\pi/3}^{\pi} r^2 \, d\theta.$$

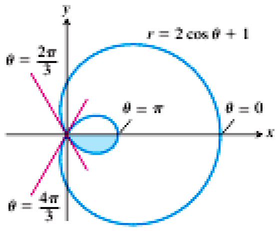



**FIGURE 10.51** The limaçon in Example 2. Limaçon (pronounced LEE-ma-sahn) is an old French word for *snail*.

Since

$$\begin{aligned} r^2 &= (2\cos\theta + 1)^2 = 4\cos^2\theta + 4\cos\theta + 1 \\ &= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1 \\ &= 2 + 2\cos 2\theta + 4\cos\theta + 1 \\ &= 3 + 2\cos 2\theta + 4\cos\theta, \end{aligned}$$

we have

$$\begin{aligned} A &= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta \\ &= \Big[3\theta + \sin 2\theta + 4\sin\theta\Big]_{2\pi/3}^{\pi} \\ &= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right) \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$

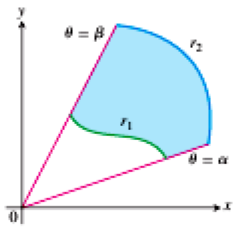



**FIGURE 10.52** The area of the shaded region is calculated by subtracting the area of the region between $r_1$ and the origin from the area of the region between $r_2$ and the origin.

To find the area of a region like the one in Figure 10.52, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2\, d\theta$ from the integral of $(1/2)r_2^2\, d\theta$. This leads to the following formula.

**Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \quad \alpha \le \theta \le \beta$**

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2\, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2\, d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$

### EXAMPLE 3 Finding Area Between Polar Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

**Solution** We sketch the region to determine its boundaries and find the limits of integration (Figure 10.53). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and $\theta$ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \\ &= 2\int_0^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry} \\ &= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \\ &= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_0^{\pi/2}\left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta \\ &= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}. \end{aligned}$$

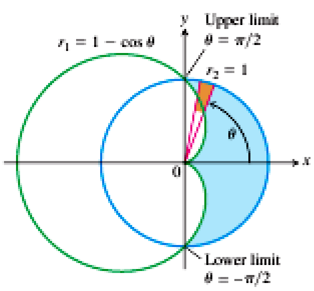



**FIGURE 10.53** The region and limits of integration in Example 3.

## Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta. \tag{2}$$

The parametric length formula, Equation (1) from Section 6.3, then gives the length as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

when Equations (2) are substituted for $x$ and $y$ (Exercise 33).

> **Length of a Polar Curve**
> If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as $\theta$ runs from $\alpha$ to $\beta$, then the length of the curve is
>
> $$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

**EXAMPLE 4** Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

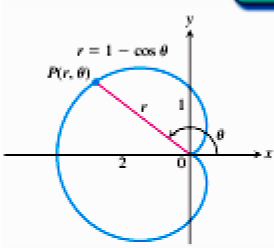



**FIGURE 10.54** Calculating the length of a cardioid (Example 4).

**Solution** We sketch the cardioid to determine the limits of integration (Figure 10.54). The point $P(r, \theta)$ traces the curve once, counterclockwise as $\theta$ runs from 0 to $2\pi$, so these are the values we take for $\alpha$ and $\beta$.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 - \cos\theta)^2 + (\sin\theta)^2 \\ &= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta \end{aligned}$$

and

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \\ &= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta \\ &= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi \\ &= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8. \end{aligned}$$