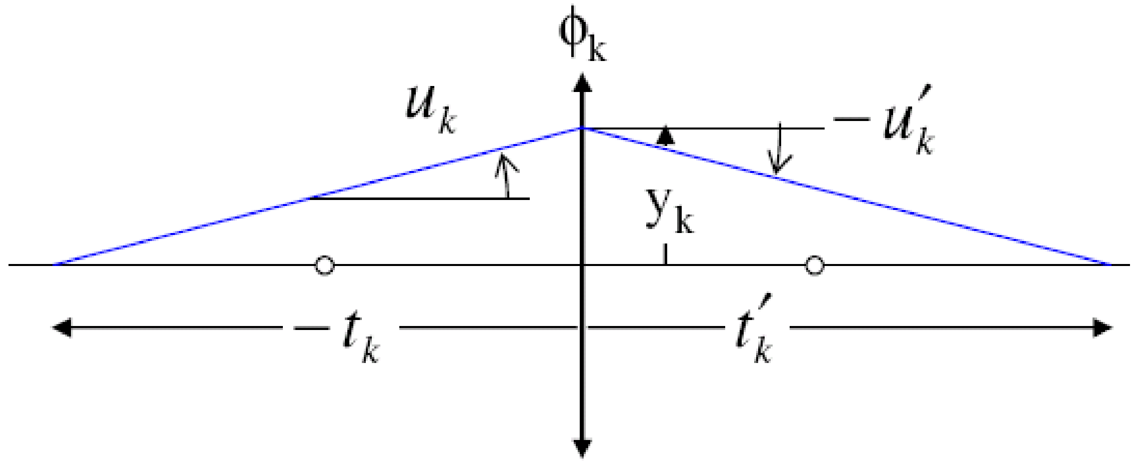
Paraxial Ray Tracing

- y-u transfer and refraction equations
  - y-u ray tracing examples
- Matrix approach to ray tracing
- System and conjugate matrices

Paraxial Ray Tracing

Derivation of refraction and transfer equations



Want to know what happens to rays as they propagate in air and interact with lens and mirrors etc.

To follow ray you have to have y (ray height) and u ray slope.

If propagating need to have distance d

If power element, need to have power of element φ

y & d in meters U, u' in radians φ In diopters

Paraxial Ray Tracing

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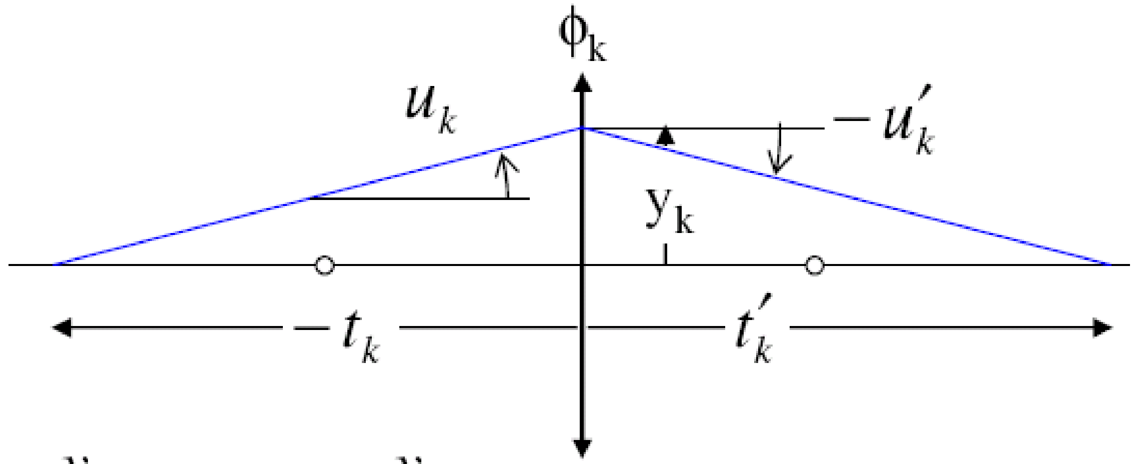
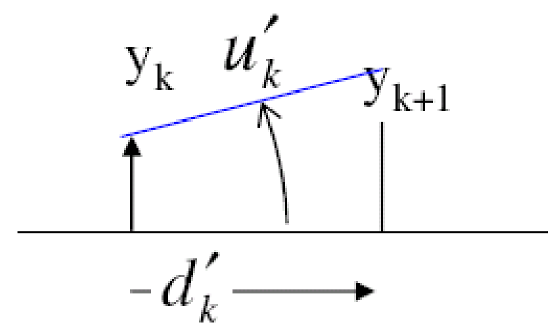


$$-t_k = \frac{y_k}{u_k}, \qquad t'_k = \frac{y_k}{-u'_k}$$ Paraxial tangents

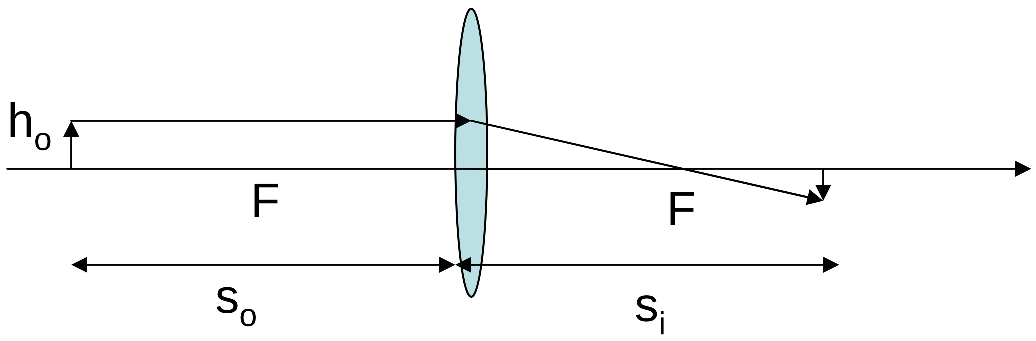
$$-\frac{u'_k}{y_k} = \phi_k - \frac{u_k}{y_k}$$ Substitute into thin-lens equation

$$\boxed{u'_k = u_k - y_k \phi_k}$$ Refraction equation



$$\boxed{y_{k+1} = y_k + u'_k d'_k}$$ Transfer equation

Simple Example

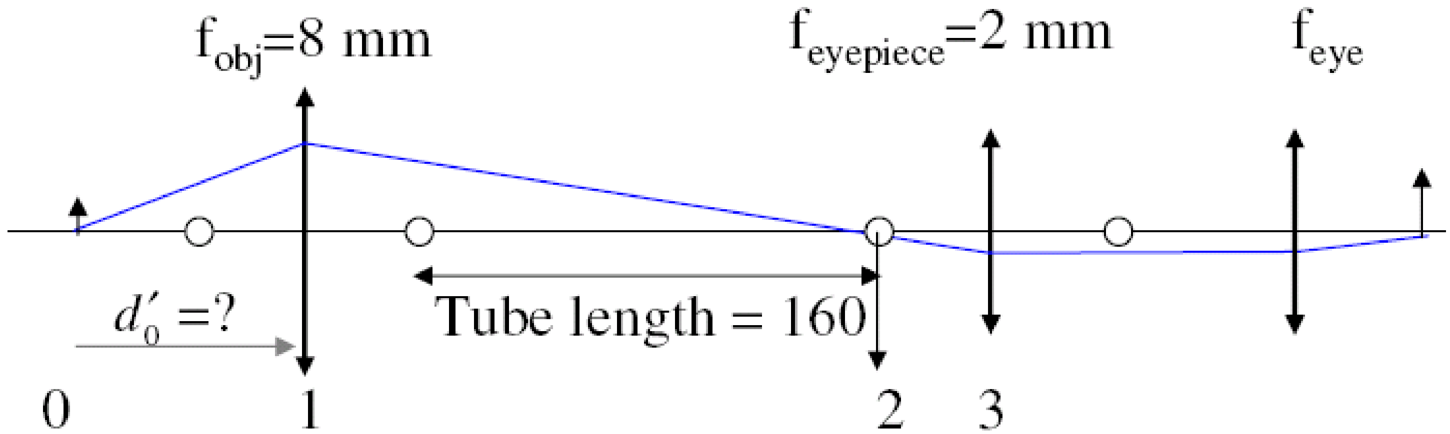


	0	1	2
φ	0	1/F	0
d	s_o	0	s_i
y (h_o)	h_o	h_o	$h_o - h_o s_i/F$
u (0)	0	$-h_o/F$	$-h_o/F$

$$y_{k+1} = y_k + u'_k d'_k \qquad u'_k = u_k - y_k \phi_k \qquad y_{k+1} = y_k + u'_k d'_k$$

y-u Tracing

Tabular method



Question to be answered: what is the front working distance of a 20x, f=8mm objective when used with a 160 mm tube, as shown?

Surface k		0	1	2	3
System	$\phi_k = 1/f$	0	1/8	0	1/2
	d_k'	8.4	168	2	
Axial ray	y_k	0	84	0	-1
	u_k'	10	-1/2	-1/2	0

Transfer: $84 = 0 + 10\,d_0'$

Dealing with different indices

thin lens

Dealing with different indices of refraction:

$$n \sin\theta = n' \sin\theta'$$ Snell's Law

$$nu \approx n'u'$$ Paraxial approximation

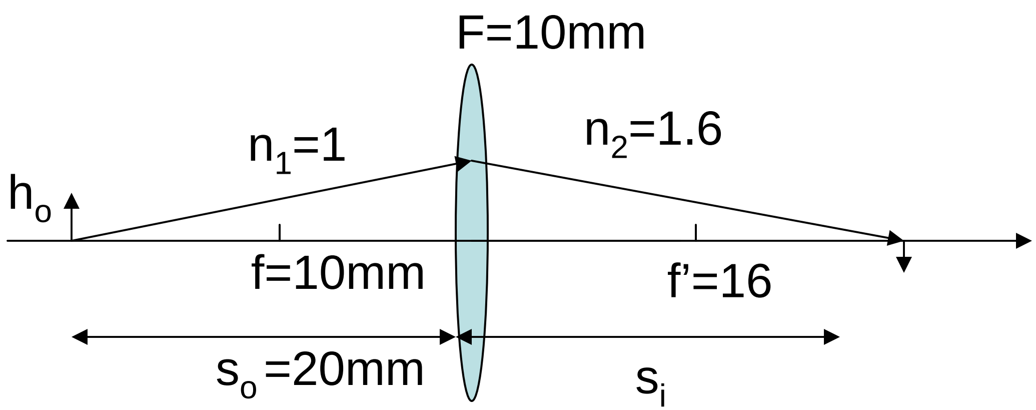
$$\boxed{\hat{u} \equiv nu}$$ Reduced angle variable

$$-\frac{n}{t} + \frac{n'}{t'} = \frac{1}{f}$$ Gaussian thin lens equation

$$\boxed{\hat{t} \equiv \frac{t}{n}}$$ Reduced distance variables

We can now write equations involving angle and distance but ignoring changes in index. Whenever we deal with problems with several difference indices, we simply make the above substitutions.

Example with different n's



	0	1	2
φ	0	1/10	0
d	S_o/n_1=20	0	S_i/n_2
y (0)	0	20	20-s_i/1.6
$\bar{u}$=nu (1)	1	1-20/10=-1	-1.6

(20-si/1.6)/-1.6=0

s_i=32mm

$$y_{k+1} = y_k + u_k' d_k'$$

$$u_k' = u_k - y_k \phi_k$$

$$y_{k+1} = y_k + u_k' d_k'$$

Check zz' = -nn'F^2(Newton's form) => z'=-1*1.6$(10)^2$/-10 = 16 => s_i=32mm

Thick lenses with different indices

Snell's Law

(1) $n_1 u_1 = n_2 u_2$

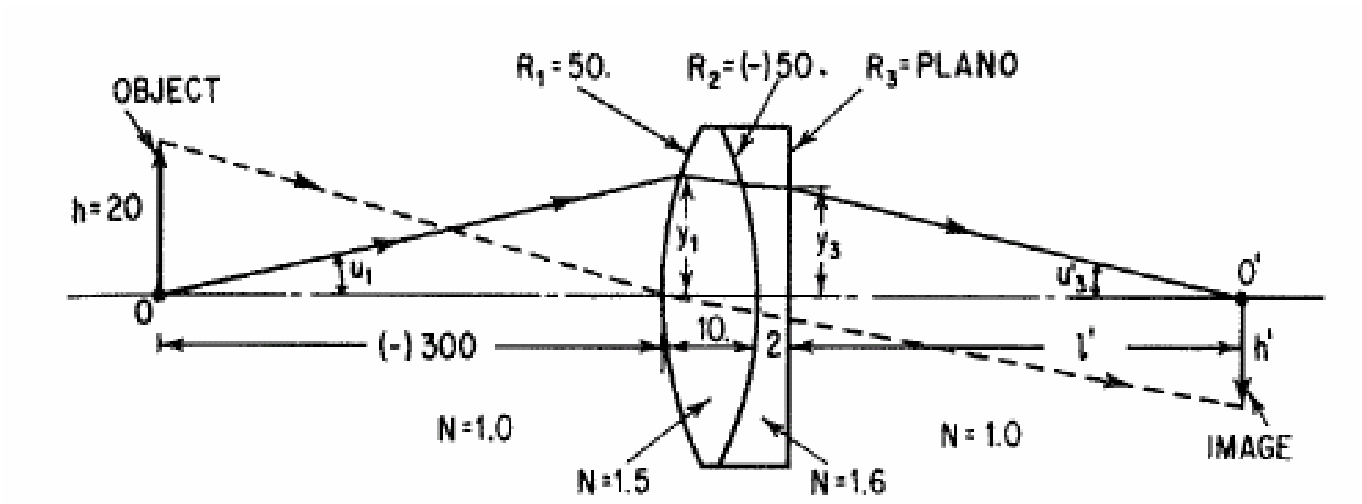
Incident on a surface with curvature of C (1/R) with index n_2 results in (Smith page 40)

(2) $n_2 u_2 = n_1 u_1 - y_1 (n_2 - n_1) C$ – refraction equation just for one surface

Propagation inside of material index n_k with thickness of t gives

(3) $y_{k+1} = y_k + t\, n_k u_k / n_k$ - normal transfer equation

y-nu Tracing
with different indices



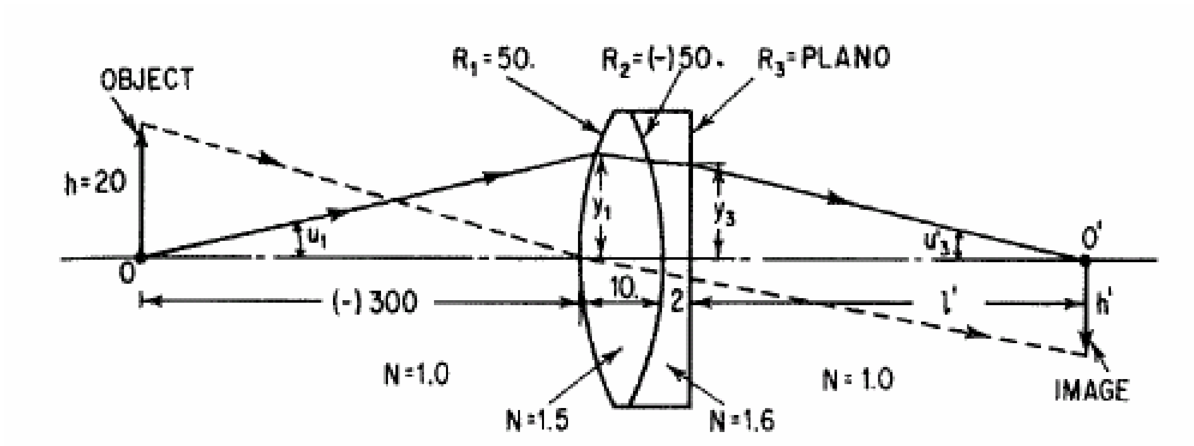
Where will the image be located ?

First write out the parameters given with the correct sign convention

h =20mm			
L1 = -300mm			n1=1.0
R1=+50mm	C1=0.02	t1=10mm	n2=1.5
R2=-50mm	C2=-0.02	t2=2mm	n3=1.6
R3 = plano	C=0		n4=1.0

y-nu Tracing

with different indices



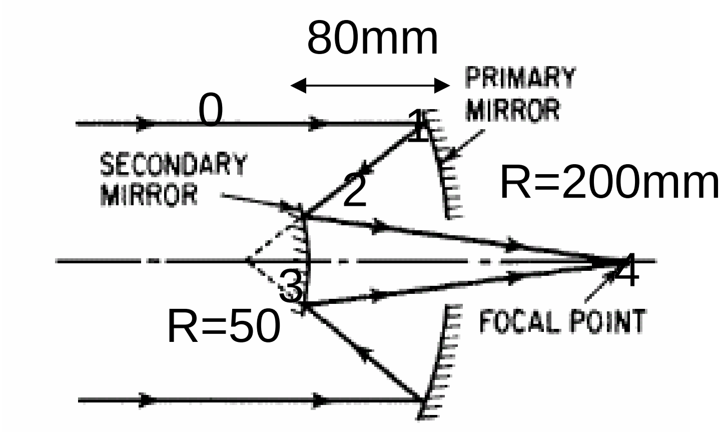
	0	1	2	3	4	5	6	7
n		1	1.5	1.5	1.6	1.6	1	1
t		300		10		2		l’
C			+0.02		-0.02		0	
Height **y**	0	10	10	9.55	9.55	9.49	9.49	0
Slope **nu**	0.333*	0.333	-0.0666	-0.066	-0.47	-0.47	-0.475	-0.475
Equation used		3	2	3	2	3	2	

l’ = -9.49/-0.475 = 199.68mm

*Arbitrary just start at O and want to find when it crosses the axis at O’

y-u for Cassegrain mirror system

Tabular method



Where is the focal point ?

	0	1	2	3	4
Radius		-200		-50	
Distance			-80		
index	1		-1		1
Ray Height	1	1		0.2	0
Slope index (nu)	0	-1/100		-0.002	

z = -0.2/-0.002 = 100mm
20mm to right of PM

Remember for mirror
$\varphi = 2n/R$

$$y_{k+1} = y_k + u'_k d'_k$$

$$u'_k = u_k - y_k \phi_k$$

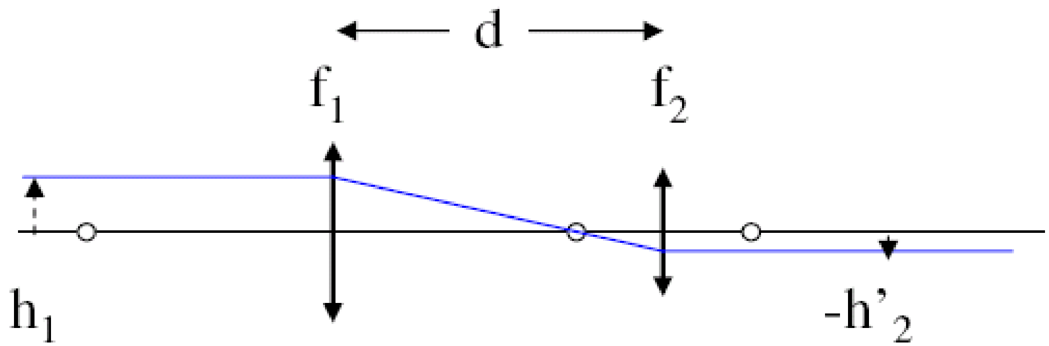
0.2 = 1+(-1)(-0.01)(-80)

-0.002=-0.01-(0.2)(0.04)

Example: The Telescope

Keplerian

Shown in the *afocal* geometry (d=f1+f2). Relaxed eye focuses at ~1m, thus telescope are usually not afocal. Analysis simpler, however.



Afocal: system has no power: ray || to OA does not intersect OA in image space

y	h_1	h_1	$h_1 - h_1(f_1+f_2)/f_1$	$h_1 - h_1(f_1+f_2))/f_1 = -h_2$
u	0	$-h_1/f_1$	$-h_1/f_1 – (h_1-h_1(f_1+f_2)/f_1)/f_2=0$	$-h_1/f_1 – (h_1-h_1(f_1+f_2)/f_1)/f_2$

$() = h_1 f_2/f_1$

$-h2 = h_1 f_2/f_1$

Independent of distance
Before and after lenses

$M = h_2/h_1 = -f_2/f_1$

$$y_{k+1} = y_k + u'_k d'_k$$

$$u'_k = u_k - y_k \phi_k$$

Example: The Telescope

Keplerian

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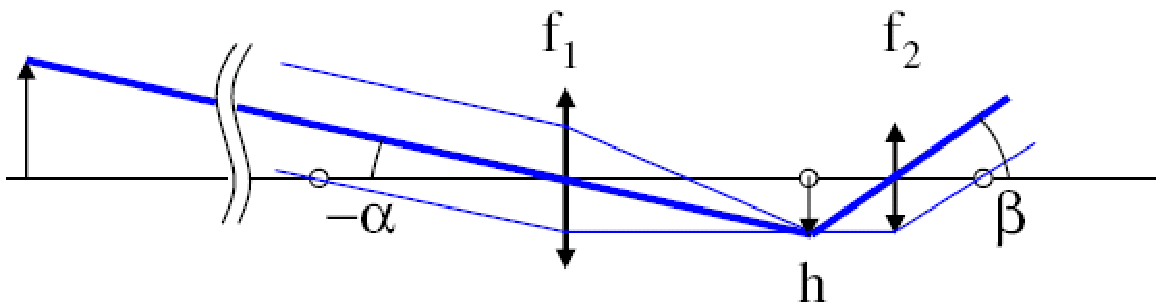


$$M_\theta \equiv \frac{\beta}{\alpha}$$ Definition of angular magnification

$$= -\frac{h/f_2}{h/f_1} = -\frac{f_1}{f_2}$$ Via similar triangles

$$= 1/M$$ This is both important and fundamental.

ABCD Matrices

Matrix formulation of ray tracing

$$\begin{bmatrix} y_k \\ u'_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\phi_k & 1 \end{bmatrix} \begin{bmatrix} y_k \\ u_k \end{bmatrix} \equiv \mathrm{R}_k \begin{bmatrix} y_k \\ u_k \end{bmatrix}$$

Refraction equation

$$u'_k = u_k - y_k \phi_k$$

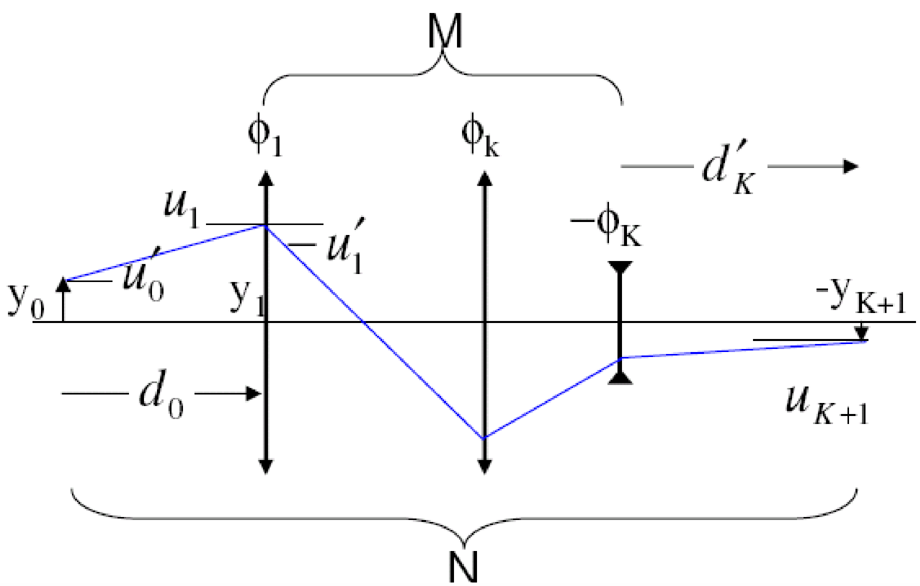
$$\begin{bmatrix} y_{k+1} \\ u_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & d'_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ u'_k \end{bmatrix} \equiv \mathrm{T}_k \begin{bmatrix} y_k \\ u'_k \end{bmatrix}$$

Transfer equation

$$y_{k+1} = y_k + u'_k d'_k$$

ABCD Matrices

Matrix formulation of ray tracing



$$\begin{bmatrix} y_K \\ u'_K \end{bmatrix} = \mathrm{R}_K \mathrm{T}_{K-1} \mathrm{R}_{K-1} \ldots \mathrm{T}_1 \mathrm{R}_1 \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} \equiv \mathrm{M} \begin{bmatrix} y_1 \\ u_1 \end{bmatrix}$$

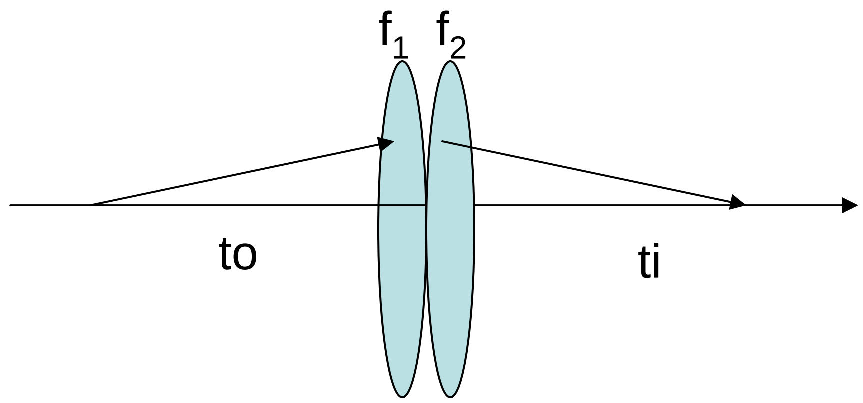
System matrix

$$\begin{bmatrix} y_{K+1} \\ u_{K+1} \end{bmatrix} = \mathrm{T}_1 \mathrm{R}_1 \mathrm{T}_0 \begin{bmatrix} y_1 \\ u'_0 \end{bmatrix} \equiv \mathrm{N} \begin{bmatrix} y_0 \\ u'_0 \end{bmatrix}$$

Conjugate matrix

Like it is one big lens

Example: 2 thin lenses in contact

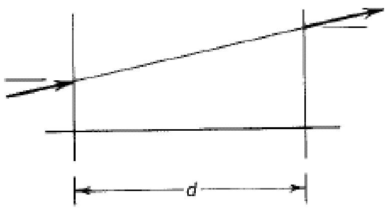


$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = T_i R_2 R_1 T_o = \begin{bmatrix} 1 & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & t_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_o \\ u_o \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 \\ -(\varphi_1 + \varphi_2) & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 - t_i(\varphi_1 + \varphi_2) & t_o + t_i - \dfrac{t_i t_o}{f_2} - \dfrac{t_i t_o}{f_1} \\ -(\varphi_1 + \varphi_2) & -\dfrac{t_o}{f_2} - \dfrac{t_o}{f_1} + 1 \end{bmatrix}$$

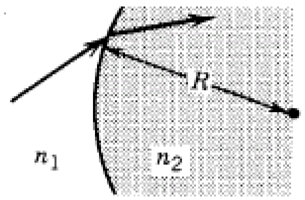
Matrices Summary

Free Space Propagation



$$\mathbf{M} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}.$$

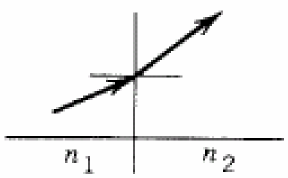
Refraction at spherical interface



Convex, $R > 0$; concave, $R < 0$

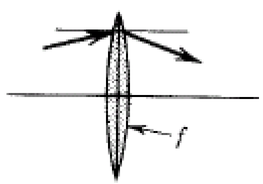
$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -\frac{(n_2 - n_1)}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix},$$

Refraction at interface



$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}.$$

Thin Lens

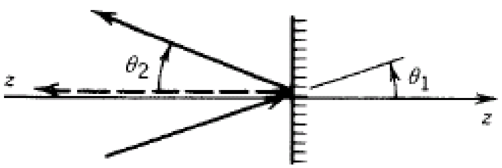


Convex, $f > 0$; concave, $f < 0$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}.$$

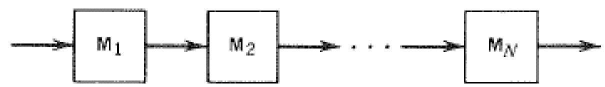
Matrices Summary

Reflection from plane mirror



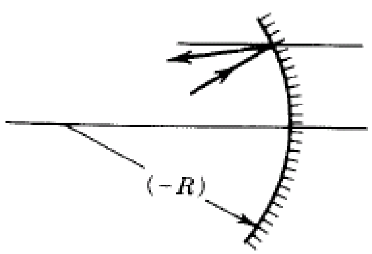
$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Cascaded optical components



$$\mathbf{M} = \mathbf{M}_N \cdots \mathbf{M}_2 \mathbf{M}_1$$

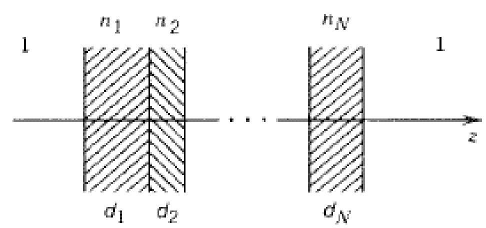
Reflection from spherical mirror



Concave, $R < 0$; convex, $R > 0$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$

Set of parallel plates



$$\mathbf{M} = \begin{bmatrix} 1 & \displaystyle\sum_{i=1}^{N} \frac{d_i}{n_i} \\ 0 & 1 \end{bmatrix}$$

Properties of M and N

$$|\mathsf{M}| = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC = 1$$

Determinant = 1

$$|\mathsf{R}| = |\mathsf{T}| = |\mathsf{M}| = |\mathsf{N}| = 1$$

Write out the matrix equation for N:

$$y_{K+1} = \mathsf{N}_{11} y_0 + \mathsf{N}_{12} u_0'$$
$$u_{K+1} = \mathsf{N}_{21} y_0 + \mathsf{N}_{22} u_0'$$

Properties of M and N

If planes 0 and K+1 are conjugates, final ray height does not depend on initial ray angle:

$N_{12} = 0$ Conjugate condition

If plane 0 is the object space focal plane, the slope at the exit plane depends only on the object height:

$N_{22} = 0$ Object at front focal plane

If plane K+1 is the image space focal plane, the image-space ray height depends only on the entrance angle:

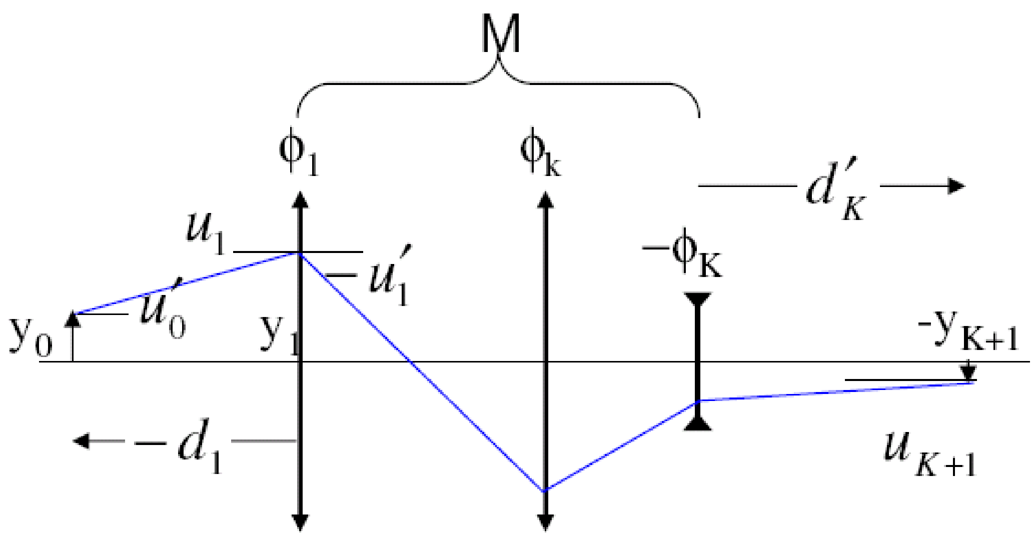
$N_{11} = 0$ Image at rear focal plane

If the system is afocal, the direction of the image-space ray depends only on the direction of the object-space ray:

$N_{21} = 0$ Afocal condition

Using M and N

Find image plane given object



Conjugate planes so $N_{12} = 0$

Using M and N matrices

$$\mathrm{N} = \mathrm{T}_{K+1}\mathrm{M}\mathrm{T}_0$$

$$= \begin{bmatrix} 1 & d'_K \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & -d_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A + d'_K C & B + d'_K D - d_1(A + d'_K C) \\ C & D - d_1 C \end{bmatrix}$$

$$= \begin{bmatrix} A + d'_K C & 0 \\ C & D - d_1 C \end{bmatrix}$$

Conjugate condition

$$d'_K = -\frac{d_1 A - B}{d_1 C - D}$$

$\mathrm{N}_{12} = 0$ gives the image location

<u>E.g. single lens</u>

$$d'_K = -\frac{d_1 1 - 0}{d_1(-\phi) - 1} \Rightarrow \frac{1}{d'_K} = \frac{1}{d_1} + \phi$$

Form of N

Effective Focal length of system or thick lens

$$M \equiv \frac{y_{K+1}}{y_0} = \mathrm{N}_{11} = A + d'_K C$$

If $\mathrm{N}_{12} = 0$ then N_{11} is the magnification

$$\mathrm{N}_{22} = \frac{1}{M}$$

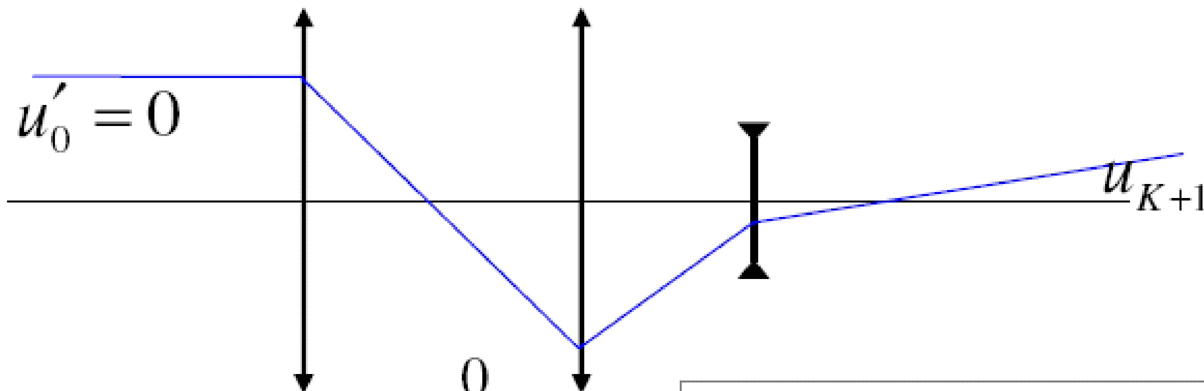
Determinant = 1

$$F = \frac{1}{\Phi} \equiv \frac{y_0}{-u_{K+1}}$$

Effective focal length & system power

Form of N

Effective Focal length of system or thick lens



$$u_{K+1} = \mathrm{N}_{21} y_0 + \cancelto{0}{\mathrm{N}_{22} u'_0}$$

$$\mathrm{N}_{21} = -\Phi$$

$$\boxed{\mathrm{N} = \begin{bmatrix} M & 0 \\ -\Phi & 1/M \end{bmatrix}}$$

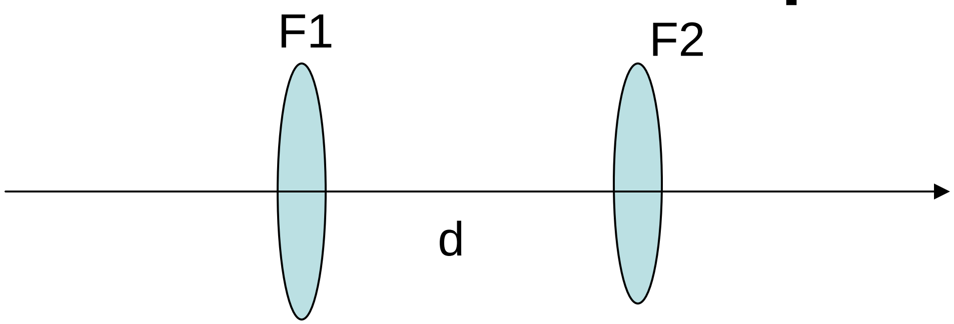
<u>E.g. single lens</u>

$$\mathrm{N} = \mathrm{T}_1 \mathrm{R}_1 \mathrm{T}_0$$

$$= \begin{bmatrix} 1-\phi t' & \cancelto{0}{t\,t'\left(-\frac{1}{t'}+\frac{1}{t}+\phi\right)} \\ -\phi & 1+\phi t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{t'}{t} & 0 \\ -\phi & \frac{t}{t'} \end{bmatrix}$$

2 lens separated by d



$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = T_4 R_3 T_2 R_1 T_0 \begin{bmatrix} y_o \\ u_o \end{bmatrix} \qquad M = R_3 T_2 R_1 \quad \text{system matrix}$$

$$M = \begin{bmatrix} 1 - d/f_2 & d \\ -1/f_1 + d/f_1 f_2 - 1/f_2 & -d/f_1 + 1 \end{bmatrix}$$

$$-M_{21} = \frac{1}{f_{eff}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

Question

Effective F of 4F system (d=f1+f2) ?

$$-M_{21} = \frac{1}{f_{eff}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

It is afocal

d> f1+f2, f is negative
d< f1+f2, f is positive

Periodic Systems

$$\begin{bmatrix} y_m \\ \theta_m \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^m \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}.$$

We can also apply the relations

$$y_{m+1} = Ay_m + B\theta_m$$

$$\theta_{m+1} = Cy_m + D\theta_m$$

Derive equations that determine the evolution of y, get rid of slope for above:

$$\theta_m = \frac{y_{m+1} - Ay_m}{B}.$$

Replacing m with $m + 1$

$$\theta_{m+1} = \frac{y_{m+2} - Ay_{m+1}}{B}.$$

Substitute these into equations above:

Periodic Systems

Yields:

$$\boxed{y_{m+2} = 2by_{m+1} - F^2 y_m,}$$

Recurrence Relation

$$y_m = y_0 h^m,$$

Where,

$$b = \frac{A + D}{2}$$

$$F^2 = AD - BC = \det[\mathbf{M}],$$

Periodic Systems

Guess a solution with initial conditions y_o and slope θ_o such that:

Put solution into recurrence relation results in: $y_m = y_0 h^m,$ For m trip around

$$h^2 - 2bh + F^2 = 0,$$

$$h = b \pm j(F^2 - b^2)^{1/2}.$$

Can define $\varphi = \cos^{-1}\frac{b}{F}$

Solution can be rewritten as:

If n1=n2, then det F=1

$$y_m = y_{\max} F^m \sin(m\varphi + \varphi_o)$$

Where $y_{\max} = y_0 / \sin\varphi_0$

Periodic Systems

$$y_m = y_{\max} F^m \sin(m\varphi + \varphi_o)$$

For the ray trajectory to be stable, |b| $\leq$ 1

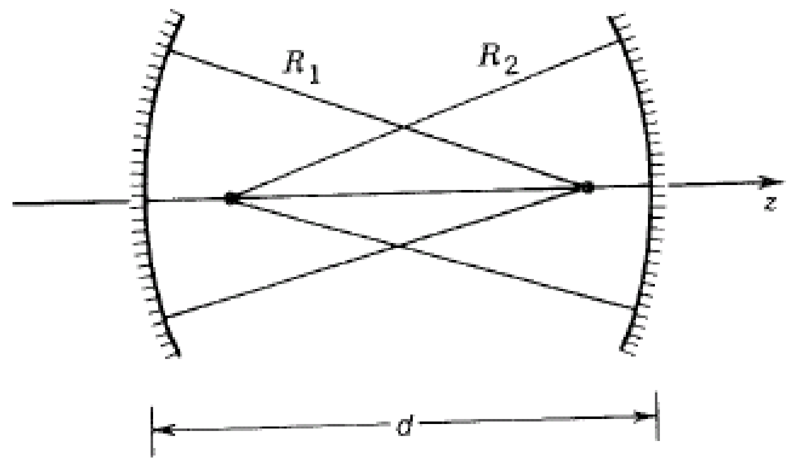
Case det F=1

Or

$$\frac{|A+D|}{2} \leq 1$$

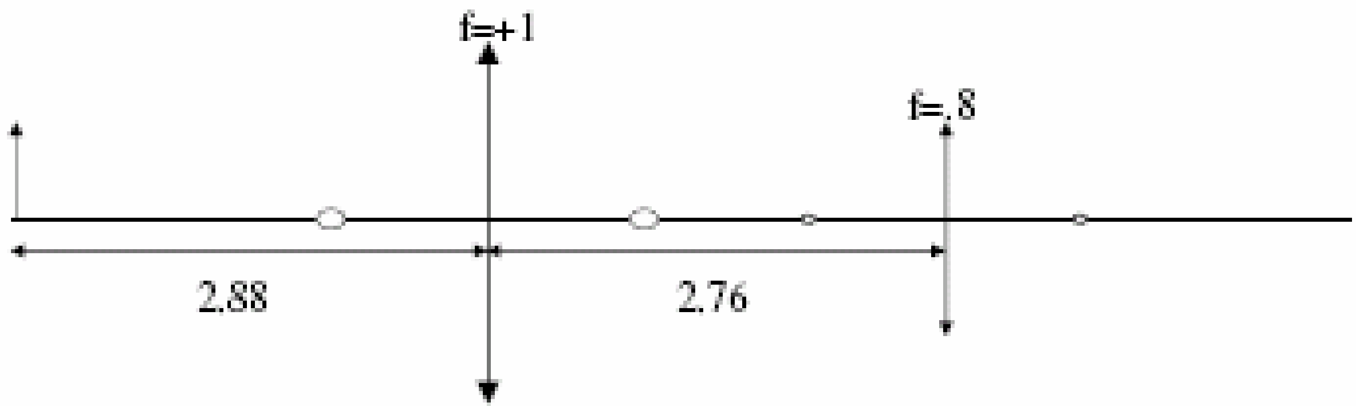
Why did we waste this time covering periodic optical systems ?

- How about resonators -> laser cavities and etalons ? They can be considered periodic optical systems. Laser cavities have to have stable ray paths for the laser to function…
- This will be a homework problem and is very important example…



Try this

Use a yu ray trace to find the image location and magnification of this system by tracing a *single* axial ray (think about how to get both those pieces of information from this ray).



Surface k		0	1	2	3
System	$\phi_k = 1/f$				
	d'_k				
Axial ray	y_k				
	u'_k				

Reading

W. Smith "Modern Optical Engineering"

Chapter 3 and Chapter 4