Lect -6 . Matrix Methods

In the beginning of the 1930s, T. Smith formulated an interesting way of handling the ray-tracing equations. The simple linear form of the expressions and the repetitive manner in which they are applied suggested the use of matrices. The processes of refraction and transfer might then be performed mathematically by matrix operators. These initial insights were not widely appreciated for almost 30 years. However, the early 1960s saw a rebirth of interest in this approach. We shall only outline some of the salient features of the method, leaving a more detailed study to the references.

Matrix Analysis of Lenses

Let's begin by writing the formulas

$$n_{t1}\alpha_{t1} = n_{i1}\alpha_{i1} - \mathscr{D}_1 y_{i1}$$

$$y_{t1} = 0 + y_{i1}$$

$$\begin{bmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{bmatrix} = \begin{bmatrix} 1 & -\mathscr{D}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n_{i1}\alpha_{i1} \\ y_{i1} \end{bmatrix}$$

We can write,

$$\boldsymbol{r}_{t1} \equiv \begin{bmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{r}_{i1} \equiv \begin{bmatrix} n_{i1}\alpha_{i1} \\ y_{i1} \end{bmatrix}$$

$$\mathscr{R}_1 \equiv \begin{bmatrix} 1 & -\mathscr{D}_1 \\ 0 & 1 \end{bmatrix}$$

refraction matrix

$$\boldsymbol{r}_{t1} = \mathscr{R}_1 \boldsymbol{r}_{i1}$$

$$n_{i2}\alpha_{i2} = n_{t1}\alpha_{t1} + 0$$

$$y_{i2} = d_{21}a_{t1} + y_{t1}$$

$$\begin{bmatrix} n_{i2}\,\alpha_{i2} \\ y_{i2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d_{21}/n_{t1} & 1 \end{bmatrix} \begin{bmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{bmatrix}$$

$$\mathscr{T}_{21} \equiv \begin{bmatrix} 1 & 0 \\ d_{21}/n_{t1} & 1 \end{bmatrix}$$

Transfer matrix

$$\boldsymbol{r}_{i2} \equiv \begin{bmatrix} n_{i2}\,\alpha_{i2} \\ y_{i2} \end{bmatrix}$$

$$\boldsymbol{r}_{i2} = \mathscr{T}_{21}\boldsymbol{r}_{t1}$$

$$\boldsymbol{r}_{i2} = \mathscr{T}_{21}\mathscr{R}_{1}\boldsymbol{r}_{i1}$$

where

$$\mathscr{R}_2 \equiv \begin{bmatrix} 1 & -\mathscr{D}_2 \\ 0 & 1 \end{bmatrix}$$

and the power of the second surface is

$$\mathscr{D}_2 = \frac{(n_{t2} - n_{i2})}{R_2}$$

From Eq. (6.26)

$$\boldsymbol{r}_{t2} = \mathscr{R}_2\mathscr{T}_{21}\mathscr{R}_1\,\boldsymbol{r}_{i1}$$

The **system matrix** $\mathscr{A}$ is then defined as

$$\mathscr{A} \equiv \mathscr{R}_2\mathscr{T}_{21}\mathscr{R}_1$$

and has the form

$$\mathscr{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Inasmuch as

$$\mathscr{A} = \begin{bmatrix} 1 & -\mathscr{D}_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ d_{21}/n_{t1} & 1 \end{bmatrix}\begin{bmatrix} 1 & -\mathscr{D}_1 \\ 0 & 1 \end{bmatrix}$$

or

$$\mathscr{A} = \begin{bmatrix} 1 & -\mathscr{D}_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\mathscr{D}_1 \\ \dfrac{d_{21}}{n_{t1}} & 1 - \dfrac{\mathscr{D}_1 d_{21}}{n_{t1}} \end{bmatrix}$$

it follows that

$$\mathscr{A} = \begin{bmatrix} 1 - \dfrac{\mathscr{D}_2 d_{21}}{n_{t1}} & -\mathscr{D}_1 - \mathscr{D}_2 + \dfrac{\mathscr{D}_2 \mathscr{D}_1 d_{21}}{n_{t1}} \\ \dfrac{d_{21}}{n_{t1}} & 1 - \dfrac{\mathscr{D}_1 d_{21}}{n_{t1}} \end{bmatrix}$$

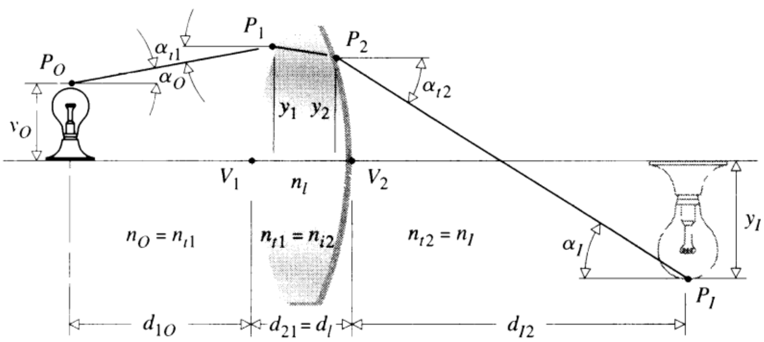
and again $|\mathscr{A}| = 1$ (Problem 6.17). Because we are working with only one lens, let's simplify the notation a little letting $d_{21} = d_l$ and $n_{t1} = n_l$. Consequently,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{\mathscr{D}_2 d_l}{n_l} & -\mathscr{D}_1 - \mathscr{D}_2 + \dfrac{\mathscr{D}_1 \mathscr{D}_2 d_l}{n_l} \\ \dfrac{d_l}{n_l} & 1 - \dfrac{\mathscr{D}_1 d_l}{n_l} \end{bmatrix} \tag{6.31}$$

The value of each element in $\mathscr{A}$ is expressed in terms of the physical lens parameters, such as thickness, index, and radii (via $\mathscr{D}$). Thus the cardinal points, which are properties of the lens determined solely by its makeup, should be deducible from $\mathscr{A}$. The system matrix in this case, Eq. (6.31), transforms an incident ray at the first surface to an emerging ray at the second surface; as a reminder, we will write it as $\mathscr{A}_{21}$.

The concept of image formation enters rather directly (Fig. 6.8) after introduction of appropriate object and image planes. Consequently, the first operator $\mathscr{T}_{1O}$ transfers the reference point from the object (i.e., P_O to P_1). The next operator $\mathscr{A}_{21}$ then carries the ray through the lens, and a final transfer $\mathscr{T}_{I2}$ brings it to the image plane (i.e., P_I). Thus the ray at the image point ($\mathbf{r}_I$) is given by

$$\mathbf{r}_I = \mathscr{T}_{I2} \mathscr{A}_{21} \mathscr{T}_{1O} \mathbf{r}_O \tag{6.32}$$



where $\mathbf{r}_O$ is the ray at P_O. In component form this is

$$\begin{bmatrix} n_I\alpha_I \\ y_I \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d_{I2}/n_I & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ d_{1O}/n_O & 1 \end{bmatrix}\begin{bmatrix} n_O\alpha_O \\ y_O \end{bmatrix} \tag{6.33}$$

Notice that $\mathcal{T}_{1O}\mathbf{r}_O = \mathbf{r}_{i1}$ and that $\mathcal{A}_{21}\mathbf{r}_{i1} = \mathbf{r}_{t2}$, hence $\mathcal{T}_{I2}\mathbf{r}_{t2} = \mathbf{r}_I$. The subscripts O, 1, 2, …, I correspond to reference points P_O, P_1, P_2, and so on, and subscripts i and t denote the side of the reference point (i.e., whether incident or transmitted). Operation by a refraction matrix will change i to t but not the reference point designation. On the other hand, operation by a transfer matrix obviously does change the latter.

Ordinarily, the physical significances of the components of $\mathcal{A}$ are found by expanding out Eq. (6.33), but this is too involved to do here. Instead, let's return to Eq. (6.31) and examine several of the terms. For example,

$$-a_{12} = \mathcal{D}_1 + \mathcal{D}_2 - \mathcal{D}_1\mathcal{D}_2 d_l/n_l$$

If we suppose, for the sake of simplicity, that the lens is in air, then

$$\mathcal{D}_1 = \frac{n_l - 1}{R_1} \quad \text{and} \quad \mathcal{D}_2 = \frac{n_l - 1}{-R_2}$$

as in Eqs. (5.70) and (5.71). Hence

$$-a_{12} = (n_l - 1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_l - 1)d_l}{R_1R_2n_l}\right]$$

But this is the expression for the focal length of a thick lens [Eq. (6.2)]; in other words,

$$a_{12} = -1/f \tag{6.34}$$

Thus the power of the lens as a whole is given by

$$-a_{12} = \mathcal{D}_l = \mathcal{D}_1 + \mathcal{D}_2 - \frac{\mathcal{D}_1\mathcal{D}_2 d_l}{n_l}$$

If the embedding media were different on each side of the lens (Fig. 6.9), this would become

$$a_{12} = -\frac{n_{i1}}{f_o} = -\frac{n_{t2}}{f_i} \tag{6.35}$$

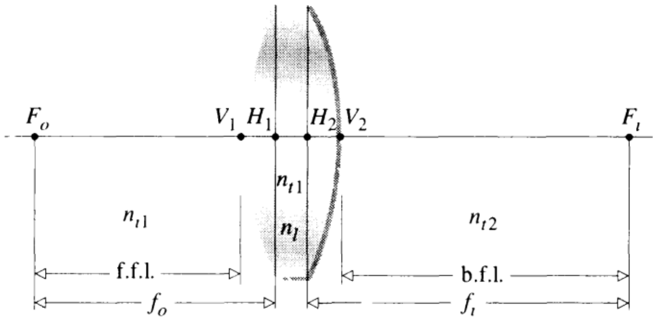


Figure 6.9 Principal planes and focal lengths.

Similarly, it is left as a problem to verify that

$$\overline{V_1H_1} = \frac{n_{i1}(1 - a_{11})}{-a_{12}} \tag{6.36}$$

and

$$\overline{V_2H_2} = \frac{n_{t2}(a_{22} - 1)}{-a_{12}} \tag{6.37}$$

which locate the principal points.

As an example of how the technique can be used, let's apply it, at least in principle, to the Tessar lens* shown in Fig. 6.10. The system matrix has the form

$$\mathscr{A}_{71} = \mathscr{R}_7\mathscr{T}_{76}\mathscr{R}_6\mathscr{T}_{65}\mathscr{R}_5\mathscr{T}_{54}\mathscr{R}_4\mathscr{T}_{43}\mathscr{R}_3\mathscr{T}_{32}\mathscr{R}_2\mathscr{T}_{21}\mathscr{R}_1$$

where

$$\mathscr{T}_{21} = \begin{bmatrix} 1 & 0 \\ \dfrac{0.357}{1.6116} & 1 \end{bmatrix} \qquad \mathscr{T}_{32} = \begin{bmatrix} 1 & 0 \\ \dfrac{0.189}{1} & 1 \end{bmatrix}$$

$$\mathscr{T}_{43} = \begin{bmatrix} 1 & 0 \\ \dfrac{0.081}{1.6053} & 1 \end{bmatrix}$$

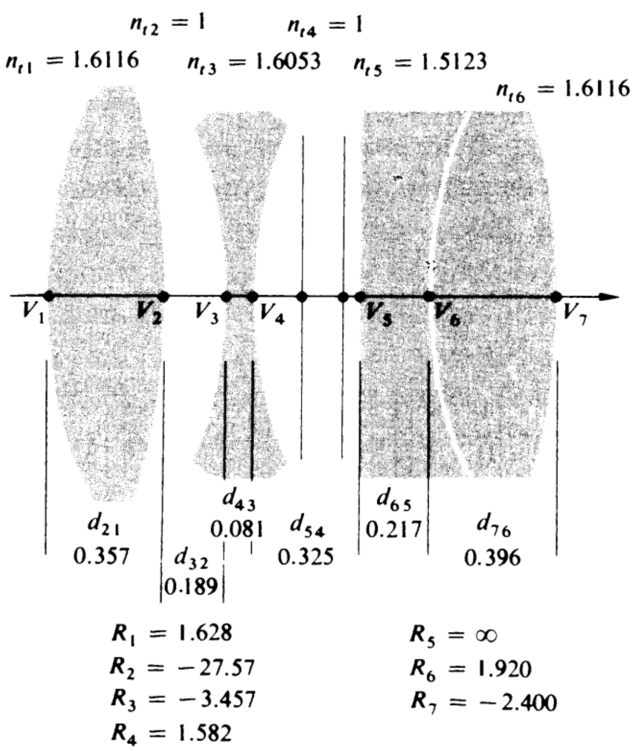


Figure 6.10 A Tessar.

and so forth. Furthermore,

$$\mathcal{R}_1 = \begin{bmatrix} 1 & -\dfrac{1.6116 - 1}{1.628} \\ 0 & 1 \end{bmatrix} \qquad \mathcal{R}_2 = \begin{bmatrix} 1 & -\dfrac{1 - 1.6116}{-27.57} \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_3 = \begin{bmatrix} 1 & -\dfrac{1.6053 - 1}{-3.457} \\ 0 & 1 \end{bmatrix}$$

and so on. Multiplying out the matrices, in what is obviously a horrendous, though conceptually simple, calculation, one presumably will get

$$\mathcal{A}_{71} = \begin{bmatrix} 0.848 & -0.198 \\ 1.338 & 0.867 \end{bmatrix}$$

and from that, $f = 5.06$, $\overline{V_1H_1} = 0.77$, and $\overline{V_7H_2} = -0.67$.

Thin Lenses

As a last point, it is often convenient to consider a system of thin lenses using the matrix representation. To that end, return to Eq. (6.31). It describes the system matrix for a single lens, and if we let dl —> 0, it corresponds to a thin lens. This is

equivalent to making $\mathscr{T}_{21}$ a unit matrix; thus

$$\mathscr{A} = \mathscr{R}_2\mathscr{R}_1 = \begin{bmatrix} 1 & -(\mathscr{D}_1 + \mathscr{D}_2) \\ 0 & 1 \end{bmatrix}$$

But as we saw in Section 5.7.2, the power of a thin lens $\mathscr{D}$ is the sum of the powers of its surfaces. Hence

$$\mathscr{A} = \begin{bmatrix} 1 & -\mathscr{D} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix}$$

In addition, for two thin lenses separated by a distance d, in air, the system matrix is

$$\mathscr{A} = \begin{bmatrix} 1 & -1/f_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/f_1 \\ 0 & 1 \end{bmatrix}$$

or

$$\mathscr{A} = \begin{bmatrix} 1 - d/f_2 & -1/f_1 + d/f_1 f_2 - 1/f_2 \\ d & -d/f_1 + 1 \end{bmatrix}$$

Clearly then,

$$-a_{12} = \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

and from Eqs. (6.36) and (6.37)

$$\overline{O_1H_1} = fd/f_2 \qquad \overline{O_2H_2} = -fd/f_1$$

all of which should be quite familiar by now. Note how easy it would be with this approach to find the focal length and principal points for a compound lens composed of three, four, or more thin lenses

Matrix Analysis of Mirrors

To derive the appropriate matrix for reflection, consult Fig. 6.11, which depicts a concave spherical mirror, and write down two equations that describe the incident and reflected rays. Again, the final form of the matrix depends on how we arrange these two equations and the signs we assign to the various quantities. What's needed is an expression relating the ray angles and another relating their heights at the point of interaction with the mirror.

First let's consider the ray angles. The Law of Reflection is $\theta_i = \theta_r$; therefore from the geometry $\tan(\alpha_i - \theta_i) = y_i/R$, and

$$(\alpha_i - \theta_i) \approx y_i/R \tag{6.38}$$

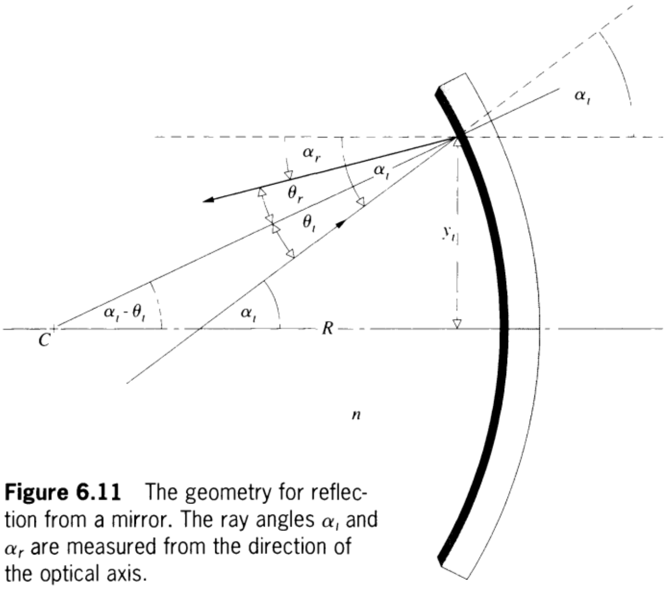


Figure 6.11 The geometry for reflection from a mirror. The ray angles α_i and α_r are measured from the direction of the optical axis.

Taking these angles to be positive, y is positive, but R isn't, and this equation will be in error as soon as we enter a negative value for the radius. Therefore rewrite it as $(\alpha_i - \theta_i) = -y_i/R$. Now to get α_r into the analysis, note that $\alpha_i = \alpha_r + 2\theta_i$ and $\theta_i = (\alpha_i - \alpha_r)/2$. Substituting this into Eq. (6.38) yields $\alpha_r = -\alpha_i - 2y_i/R$, and multiplying by n, the index of the surrounding medium (where usually $n = 1$), leads to

$$n\alpha_r = -n\alpha_i - 2ny_i/R$$

The second necessary equation is simply $y_r = y_i$ and so

$$\begin{bmatrix} n\alpha_r \\ y_r \end{bmatrix} = \begin{bmatrix} -1 & -2n/R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n\alpha_i \\ y_i \end{bmatrix}$$

Thus the mirror matrix $\mathcal{M}$ for a spherical configuration is given by

$$\mathcal{M}_\circ = \begin{bmatrix} -1 & -2n/R \\ 0 & 1 \end{bmatrix} \tag{6.39}$$

remembering from Eq. (5.49) that $f = -2/R$.

Flat Mirrors and the Planar Optical Cavity

For a flat mirror ($R \to \infty$) in air ($n = 1$), the matrix is

$$\mathcal{M}_| = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

where the minus sign in the first position reverses the ray upon reflection. Figure 6.12 shows two planar mirrors facing each other forming an **optical cavity** (p. 591). Light leaves point *O*, traverses the gap in the positive direction, is reflected by mirror-1, retraces the gap in the negative direction, and is reflected by mirror-2. The system matrix is

$$\mathcal{A} = \mathcal{M}_{|2}\mathcal{T}_{21}\mathcal{M}_{|1}\mathcal{T}_{12}$$

$$\mathcal{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}$$

and
$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 2d & 1 \end{bmatrix}$$

where again the determinant of the system matrix is one: $|\mathcal{A}| = 1$. Presumably, if the initial ray was axial ($\alpha = 0$), the system matrix should bring it back to its starting point so that the final ray $\mathbf{r}_f$ is identical to the initial ray $\mathbf{r}_i$. That is,

$$\mathcal{A}\mathbf{r}_i = \mathbf{r}_f = \mathbf{r}_i$$

This is a special kind of mathematical relationship known as an ***eigenvalue equation*** where, a bit more generally,

$$\mathcal{A}\mathbf{r}_i = a\mathbf{r}_i$$

and a is a constant. In other words,

$$\begin{bmatrix} 1 & 0 \\ 2d & 1 \end{bmatrix}\begin{bmatrix} \alpha_i \\ y_i \end{bmatrix} = a\begin{bmatrix} \alpha_i \\ y_i \end{bmatrix}$$

If $\alpha_i = 0$ and the initial ray is launched axially, then $y_i = ay_i$

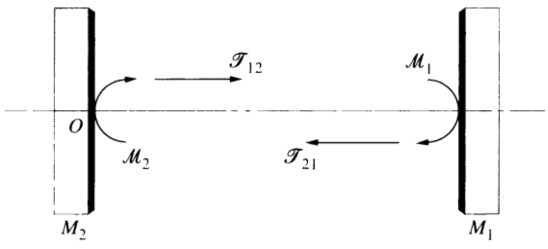


Figure 6.12 A schematic representation of a planar cavity formed by mirrors M_1 and M_2.

and it follows that $a = 1$. The system matrix functions like a unit matrix that carries $\boldsymbol{r}_i$ into $\boldsymbol{r}_i$ after two reflections. Axial rays of light travel back and forth across the so-called ***resonant cavity*** without escaping.

Cavities can be constructed in a number of different ways using a variety of mirrors (Fig. 13.12, p. 594). If after traversing a cavity some number of times the light ray returns to its original location and orientation, the beam will be trapped and the cavity is said to be *stable*; that's why the eigenvalue discussion is important. To analyze the *confocal cavity* composed of two concave spherical mirrors facing each other, see Problem 6.24.