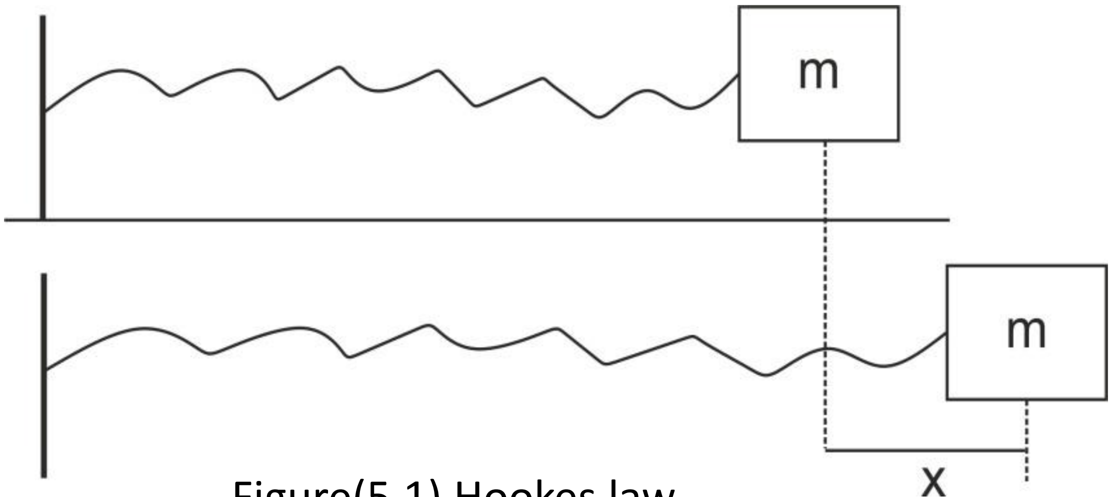
Harmonic Oscillator

Hermit's differential equation



Figure(5.1) Hookes law

$f = -kx$ .....(5.1)

$f = ma = m\frac{d^2x}{dt^2}$ ....(5.2)

$m\frac{d^2x}{dt^2} = -kx$

- $m\frac{d^2x}{dt^2} + kx = 0 \ldots\ldots\ldots (5.3)$
- $\frac{d^2x}{dt^2} = +\frac{k}{m}x = 0 \ldots\ldots\ldots.\ldots (2)$
- $\omega^2 = \frac{k}{m}$
- $\frac{d^2x}{dt^2} + \omega^2 x = 0 \ldots\ldots..\ldots... (5.4)$
- $x = A\cos\omega t + B\ \sin\omega t$

- $-\frac{\hbar^2}{2m}\ \frac{d^2\psi}{dx^2} + v\psi = E\psi$
- $dv = \ -f dx = \ -(-\ kx)\ dx = kxdx$
- $v = \int dv = \int kxdx = \frac{1}{2}\ kx^2$
- $-\frac{\hbar^2}{2m}\ \frac{d^2\psi}{dx^2} + \frac{1}{2} kx^2\psi = E\psi$
- $\frac{d^2\psi}{dx^2} + \left(-\frac{2m}{\hbar^2}\ \ \frac{1}{2} kx^2\right)\psi = \ -\frac{2m\,\mathrm{E}}{\hbar^2}\ \psi$
- $\frac{d^2\psi}{dx^2} + \left(-\frac{mkx^2}{\hbar^2}\right)\psi = \ -\frac{2m\,\mathrm{E}}{\hbar^2}\ \psi$
- $\frac{d^2\psi}{dx^2} + \left(\frac{2m\,\mathrm{E}}{\hbar^2} - \frac{mkx^2}{\hbar^2}\right)\psi = 0$

- Let $y = \propto x \;\; \therefore \frac{dy}{dx} = \propto$ and $x = \frac{y}{\propto}$
- $\frac{d\psi}{dx} = \frac{d\psi}{dy}\,\frac{dy}{dx}$
- But $\frac{dy}{dx} = \propto$
- $\therefore \frac{d\psi}{dx} = \propto \frac{d\psi}{dy}$
- $\frac{d}{dx} = \propto \frac{d}{dy}$ & $\frac{d^2}{dx^2} = \propto^2 \;\frac{d^2}{dy^2}$
- $\propto^2 \;\frac{d^2\psi}{dy^2} + \left(\frac{2m\,\mathrm{E}}{\hbar^2} - \frac{mky^2}{\hbar^2 \propto^2}\right)\psi = 0$
- $\frac{d^2\psi}{dy^2} + \left(\frac{2m\,\mathrm{E}}{\hbar^2 \propto^2} - \frac{mky^2}{\hbar^2 \propto^4}\right)\psi = 0$

- $\therefore \propto^4 = \frac{mk}{\hbar^2}$ & $\propto^2 = \sqrt{\frac{mk}{\hbar^2}}$
- $\omega^2 = \frac{k}{m} \longrightarrow k = m\omega^2$
- $\propto^2 = \sqrt{\frac{m^2\omega^2}{\hbar^2}}$ ……………. (5.5)
- $\propto^2 = \frac{m\omega}{\hbar}$
- $\frac{d^2\psi}{dy^2} + \left(\frac{2m\,\mathrm{E}}{\hbar^2\frac{m\omega}{\hbar}} - \frac{mky^2}{\hbar^2\,\frac{mk}{\hbar^2}}\right)\psi = 0$

- $\frac{d^2\psi}{dy^2} + \left(\frac{2\text{E}}{\hbar\omega} - y^2\right)\ \psi = 0$
- Let $\frac{2\text{E}}{\hbar\omega} = \text{y}$
- $\therefore \frac{d^2\psi}{dy^2} +\ (\text{y} - y^2)\,\psi = 0$ …..(5.6)
- (Hermit differential equation for harmonic oscillator)

- **The energy of Harmonic Oscillator**
- $E_n = \frac{(2n+1)\hbar\omega}{2} = \left(n + \frac{1}{2}\right)\hbar\omega$……(5.7)
- n=0 for ground state n=0, 1, 2, 3…
- Then $E_0 = \frac{1}{2}\ \hbar\omega(zero\ point\ energy)$ …..(5.8)
- $E_1 = \frac{3}{2}\ \hbar\omega first\ excited\ state, E_2 = \frac{5}{2}\hbar\omega\ second\ excited\ state.$

- **The wave function of Harmonic Oscillator**
- $\psi_n = N_n\, H_n(\propto x)\, e^{\frac{-\propto^2 x^2}{2}}$ …….(5.9)
- $\propto^2 = \frac{m\omega}{\hbar}$
- **Where:**
- N_n: Normalized constant.
- H_n: Harmonic function.
- $\propto x$: Sometimes equals to Z.
- $N_n = \sqrt{\frac{\propto}{\sqrt{\pi}\, 2^n\, n!}}$, $0! = 1$ ……(5.10)
- $H_n(z) = (-1)^n{}_e\, z^2 \frac{d^n}{dz^n}\, e^{-z^2}$ …..(5.11)

- **Solved problems:**
- 1-Find the wave function of the harmonic Oscillator in the ground state?
- **Solution:**
- $\psi_n = N_n \, H_n \, (\propto x) \, e^{\frac{-\propto^2 x^2}{2}}$
- $N_n = \sqrt{\frac{\propto}{\sqrt{\pi} \, 2^n \, n!}}$
- $N_0 = \sqrt{\frac{\propto}{\sqrt{\pi} \, 2^0 \, 0!}} = \left(\frac{\propto}{\sqrt{\pi}}\right)^{\frac{1}{2}}$
- $\mathrm{H}_n(z) = (-1)^n{}_e \;\; z^2 \, \frac{d^n}{dz^n} \, e^{-z^2}$
- $\mathrm{H}_n(z) = (-1)^o{}_e \, z^2 \, \frac{d^0}{dz^0} \, e^{-z^2}$
- $= e^{z^2} \, e^{-z^2} = e^0 = 1 \qquad\qquad (z = \propto x)$
- $\therefore \psi_0 = \left(\frac{\propto}{\sqrt{\pi}}\right)^{\frac{1}{2}}{}_e \; - \frac{\propto^2 x^2}{2}$

- **2-**Find the wave function of the harmonic Oscillator in the first excites state?
- **Solution:**
- $\psi_n = N_n \, H_n \,(\propto x) \, e^{\frac{-\propto^2 x^2}{2}}$
- $N_n = \sqrt{\frac{\propto}{\sqrt{\pi}\, 2^n \, n!}}$
- $N_1 = \sqrt{\frac{\propto}{\sqrt{\pi}\, 2^1 \, 1!}} = \left(\frac{\propto}{2\sqrt{\pi}}\right)^{\frac{1}{2}}$
- $\mathrm{H}_n(\propto x) = (-1)^n{}_e \;\, z^2 \;\frac{d^n}{dz^n}\; e^{-z^2}$
- $\mathrm{H}_n(z) = (-1)^1{}_e \, z^2 \;\frac{d}{dz}\; e^{-z^2} = e^{-z^2}(-2Z)e^{-z^2} = 2Z = 2 \propto x$
- $\therefore \psi_1(\propto x) = \left(\frac{\propto}{2\sqrt{\pi}}\right)^{\frac{1}{2}} \, 2 \propto x \; e^{\frac{-\propto^2 x^2}{2}}$

- **Problems:**
- -find ψ_2 for Harmonic Oscillator.
- prove $\psi_4(x) = \sqrt{\frac{\propto}{384\sqrt{\pi}}}(16 \propto^4 x^4 - 48 \propto^2 x^2 + 12)e^{\frac{-\propto^2 x^2}{2}}$
- \- prove $\psi_5(x) = \sqrt{\frac{\propto}{3840\sqrt{\pi}}}(32 \propto^5 x^5 - 160 \propto^3 3 + 120 \propto x)e^{\frac{-\propto^2 x^2}{2}}$
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- *** Solved problem:** State the wave function for the harmonic Oscillator at the 3rd excited state and calculate its energy
- **solution:**
- $\psi_n(\propto x)\, N_n\, H_n(\propto x)\, e^{\frac{-\propto^2 x^2}{2}}$ $\qquad \propto^2 = \sqrt{\frac{mk}{\hbar}} = \frac{mw}{\hbar}$
- $N_n = \sqrt{\frac{\propto}{\sqrt{\pi}\, 2^n\, n!}}$
- $\therefore N_3 = \sqrt{\frac{\propto}{\sqrt{\pi}\, 2^3\, 3!}} = \left(\frac{\propto}{48\sqrt{\pi}}\right)^{\frac{1}{2}}$
- $\mathrm{H}_n(z) = (-1)^n\, e^{z^2} \frac{d^n}{dz^n} e^{-z^2}$ Where $z = \propto x$
- $\mathrm{H}_3(z) = (-1)^3\, e^{z^2} \frac{d^3}{dz^3} e^{-z^2}$
- $H_3(\propto x) = 8 \propto^3 x^3 - 12 \propto x$
- $\therefore \psi_{3(\propto x)} = \left(\frac{\propto}{48\sqrt{\pi}}\right)^{\frac{1}{2}} (8 \propto^3 x^3 - 12 \propto x)\, e^{\frac{-\propto^2 x^2}{2}}$, En= $(n+\frac{1}{2})\, \hbar\omega$, E3= $\frac{7}{2}\, \hbar\omega$