$$e^{\mp x} = 1 \mp \frac{x}{1!} + \frac{x^2}{2!} \mp \frac{x^3}{3!} + \frac{x^4}{4!} \mp \frac{x^5}{5!} + \text{ ....Odd function}$$

$$e^{\mp x^2} = 1 \mp \frac{x^2}{2!} \mp \frac{x^4}{4!} + \frac{x^6}{3!} \mp \frac{x^8}{4!} + \frac{x^{10}}{5!} - \text{ ....Even function}$$

$$e^{-\propto^2 x^2} = 1 - \frac{\propto^2 x^2}{1!} + \frac{\propto^4 x^4}{2!} - \frac{\propto^6 x^6}{3!} + \frac{\propto^8 x^8}{4!} + .. \text{ Even function}$$

$\int_{-\infty}^{\infty} f\,(x)dx = 0$ For odd function

For example: $\int_{-2}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{-2}^{2} = \frac{4}{2} - \frac{4}{2} = 0$

$\int_{-\infty}^{\infty} f\,(x)dx = 2\int_{0}^{\infty} f\,(x)dx$ For even function

$$\int_{-1}^{1} 5x^4\,dx = 5\left.\frac{x^5}{5}\right|_{-1}^{1} = 2$$

$$= 2\int_{0}^{1} 5x^4 = 2\left[\frac{5x^5}{5}\right]_{0}^{1} = 2\,(1)^5 = 2$$

- $\int_0^\infty x^{2n}\, e^{-ax^2}\, dx = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}}$ ....(5.12)
- **For example:** $\int_0^\infty e^{-ax^2}\, dx, n = 0 = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}}$
- $= \frac{(2(0)-1)!!}{2^{0+1}} \sqrt{\frac{\pi}{a^{2(0)+1}}} = \frac{1}{2} \sqrt{\frac{\pi}{a}}$
- $\int_0^\infty x^2\ e^{-ax^2} dx, 2n = 2 \implies n = 1$
- $\frac{(2(1)-1)!!}{2^{1+1}} \sqrt{\frac{\pi}{a^{2+1}}} = \frac{1}{4} \sqrt{\frac{\pi}{a^3}}$
- $\int_0^\infty e^{-\propto x^2}\, dx = \ \ n = 0\ \frac{(2(0)-1)!!}{2^{0+1}} \sqrt{\frac{\pi}{(\propto^2)^{2(0)+1}}} (a = \propto^2)$
- $= \frac{1}{2} \sqrt{\frac{\pi}{\propto^2}}$
- $\int_0^\infty x^4\ e^{-\propto^2 x^2} dx \implies 2n = 4 \implies n = 2$
- $\frac{(2(2)-1)!!}{2^{2+1}} \sqrt{\frac{\pi}{(\propto^2)^{2(2)+1}}} = \frac{3}{8} \sqrt{\frac{\pi}{\propto^{10}}}$

- $\int_0^\infty x^6\, e^{-\propto x^2}\, dx =$
- $2n = 6 \Longrightarrow n = 3$
- $\frac{(2(3)-1)!!}{2^{3+1}} \sqrt{\frac{\pi}{(\propto^2)^{2(3)+1}}}$
- $= \frac{5!!}{16} \sqrt{\frac{\pi}{\propto^{14}}} = \frac{15}{16} \sqrt{\frac{\pi}{\propto^{14}}}$

- **Normalized condition of harmonic oscillator**
- $\int_{-\infty}^{\infty} \psi_n \psi_n * dx = 1$ ....(5.14)
- **Solved problem:** prove the wave function of the harmonic Oscillator at the 1st excited state is normalized.
- **Solve:**
- $\psi_n(x)\ N_n\ H_n(\propto x)\ e^{\frac{-\propto^2 x^2}{2}}$
- $N_n = \sqrt{\frac{\propto}{\sqrt{\pi}\ 2^n\ n!}} \Longrightarrow N_1 = \sqrt{\frac{\propto}{\sqrt{\pi}\ 2!}} = \left(\frac{\propto}{2\sqrt{\pi}}\right)^{\frac{1}{2}}$
- $\mathrm{H}_n(z) = (-1)^n\ e^{z^2}\ \frac{d^n}{dz^n}\ e^{-z^2}$

- $\mathrm{H}_1(z) = (-1)^1 \; e^{z^2} \; \frac{d}{dz} \; e^{-z^2}$
- $\mathrm{H}_1(z) = \; - e^{z^2}\left(-2Ze^{-z^2}\right) = 2Z$
- $\mathrm{H}_1(\propto x) = 2 \propto x$
- $\therefore \psi_1 \; = \left(\frac{\propto}{2\sqrt{\pi}}\right)^{\frac{1}{2}} 2 \propto x \; e^{\frac{-\propto^2 x^2}{2}}$
- $\int_{-\infty}^{\infty} \psi_n \; \psi_n^* \; dx = 1$
- $\int_{-\infty}^{\infty} \left(\frac{\propto}{2\sqrt{\pi}}\right)^{\frac{1}{2}} 2 \propto x \; e^{\frac{-\propto^2 x^2}{2}} \quad . \; \left(\frac{\propto}{\sqrt[2]{\pi}}\right)^{\frac{1}{2}} 2 \propto x \; e^{\frac{-\propto^2 x^2}{2}} dx$
- $\frac{\propto^3}{2\sqrt{\pi}} \int_{????}^{\infty} x^2 e^{-\propto x^2} \; dx = 2 \, \frac{\propto}{\sqrt[2]{\pi}} \; \int_0^{\infty} x^2 e^{-\propto x^2}$
- $\int_0^{\infty} x^{2n} e^{-ax^2} \; dx = \frac{(2n-1)!!}{2^{n+1}} \; \sqrt{\frac{\pi}{(a)^{2n+1}}}$
- $2n = 2 \Longrightarrow n = 1$
- $\int_0^{\infty} x^2 e^{-2x^2} \; dx = \frac{(2(1)-1)!!}{2^{1+1}} \; \sqrt{\frac{\pi}{(\propto^2)^{2(1)+1}}}$
- $= \; \frac{1}{4} \; \sqrt{\frac{\pi}{\propto^6}}$

- $\therefore$ Total integration $\frac{4\propto^3}{\sqrt{\pi}}\frac{1}{4}\sqrt{\frac{\pi}{\propto^6}}$
- $\frac{4\propto^3}{\sqrt{\pi}}\frac{1}{4}\sqrt{\frac{\pi}{\propto^6}} = 1$
- $\therefore$ The function is normalized
- **Solved problem:** Prove ψ_2 for Harmonic Oscillator is normalized.
- **solution:**
- $\psi_2 = \sqrt{\frac{\propto}{8\sqrt{\pi}}}(4\propto^2 x^2 - 2)\, e^{\frac{-\propto^2 x^2}{2}} = \psi_2^*$
- $\int_{-\infty}^{\infty} \psi_2^* \ \psi_2 \, dx = 1$
- $\int_{-\infty}^{\infty} \sqrt{\frac{\propto}{8\sqrt{\pi}}}\ (4\propto^2 x^2 - 2)\, e^{\frac{-\propto^2 x^2}{2}} \qquad \sqrt{\frac{\propto}{8\sqrt{\pi}}}(4\propto^2 x^2 - 2)\, e^{\frac{-\propto^2 x^2}{2}}\, dx$
- $= 2.\frac{\propto}{8\sqrt{\pi}} \int_0^{\infty}(4\propto^2 x^2 - 2)^2\, e^{-\propto^2 x^2}\, dx$
- $\frac{\propto}{4\sqrt{\pi}} \int_0^{\infty}(16\propto^4 x^4 - 16\propto^2 x^2 + 4)e^{-\propto^2 x^2}\, dx$
- $\frac{\propto}{4\sqrt{\pi}} \int_0^{\infty}(16\propto^4 x^4)e^{-\propto^2 x^2}\, dx - \frac{\propto}{4\sqrt{\pi}}\int_0^{\infty}(16\propto^2 x^2)e^{-\propto^2 x^2}\, dx + \frac{\propto}{4\sqrt{\pi}} \int_0^{\infty} 4e^{-\propto^2 x^2}\, dx$
- $\frac{4\propto^5}{\sqrt{\pi}}\ \frac{3}{8}\sqrt{\frac{\pi}{\propto^{10}}} - \frac{4\propto^3}{\sqrt{\pi}}\ \frac{1}{4}\sqrt{\frac{\pi}{\propto^6}} + \frac{\propto}{\sqrt{\pi}}\ \frac{1}{2}\sqrt{\frac{\pi}{\propto^2}} = 1$
- $\therefore$ ψ_2 is normalized

The orthogonal OF harmonic oscillator

- $\int_{-\infty}^{\infty} \psi_{odd}^{*}\ \psi_{even}\ dx = 0$ ......(5.15)
- $\int_{-\infty}^{\infty} \psi_{m}^{*}\ \psi_{n}\ dx = 0$ The orthogonal condition
-
- **Solved problem:**
- Prove ψ_0 and ψ_2 for Harmonic Oscillator are orthogonal.
- **solution:**
- $\psi_0 = \left(\frac{\propto}{\pi}\right)^{\frac{1}{2}}\ e^{\frac{-\propto^2 x^2}{2}}$
- $\psi_2 = \left(\frac{\propto}{8\sqrt{\pi}}\right)^{\frac{1}{2}}\ (4 \propto^2 x^2 - 2) e^{\frac{-\propto^2 x^2}{2}}$
- $\int_{-\infty}^{\infty} \left(\frac{\propto}{\pi}\right)^{\frac{1}{2}}\ e^{\frac{-\propto^2 x^2}{2}}\ \left(\frac{\propto}{8\sqrt{\pi}}\right)^{\frac{1}{2}} (4 \propto^2 x^2 - 2) e^{\frac{-\propto^2 x^2}{2}} dx$
- $\frac{\propto}{2\sqrt{2\pi}}\ \int^{\infty} (4 \propto^2 x^2 - 2) e^{-\propto^2 x^2}\ dx$
- $\frac{\propto}{\sqrt{2\pi}}\ \int_0^{\infty} \left(4 \propto^2 x^2\ e^{-\propto^2 x^2}\right) dx - \frac{\propto}{\sqrt{2\pi}}\ \int_0^{\infty} 2\ e^{-\propto^2 x^2}\ dx$
- $\frac{4\propto^3}{\sqrt{2\pi}}\ \frac{1}{4} \sqrt{\frac{\pi}{\propto^6}} - \frac{2\propto}{\sqrt{2\pi}}\ \frac{1}{2} \sqrt{\frac{\pi}{\propto^2}}$
- $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = zero \Longrightarrow \psi_0\ and\ \psi_2$ are orthogonal

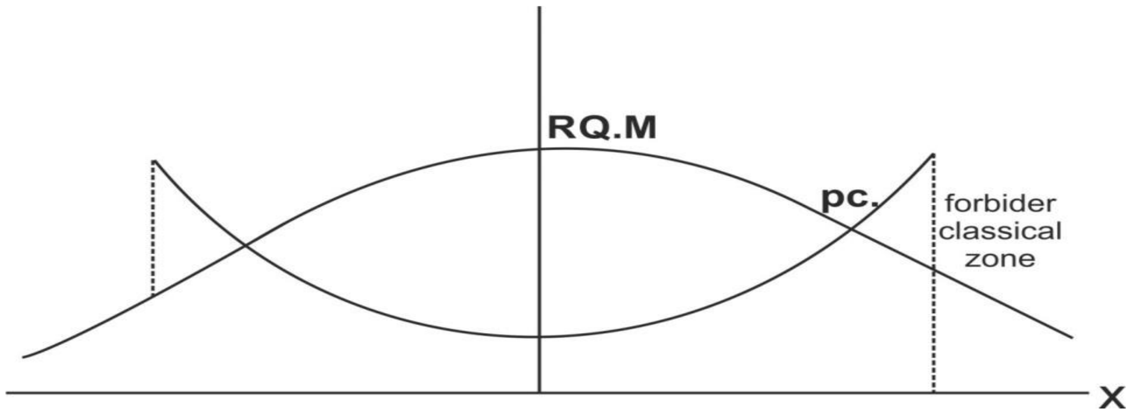
-
- *** Problems:**
- \- Prove ψ_1 and ψ_3 for Harmonic Oscillator are orthogonal.
- \- Prove ψ_3 is normalized.
- \- Prove ψ_2 and ψ_4 are orthogonal.
- \- Prove ψ_4 is normalized.
- \- Prove ψ_1 and ψ_5 are orthogonal.

- Example : **compare between ψ_4 and ψ_5 for Harmonic Oscillator**
- **Solution:**

ψ_4	ψ_5
$E_n = \frac{9}{2}\hbar w$ less than ψ_5	$E_s = \frac{11}{2}\hbar w$ greater than ψ_4
$N_n = \sqrt{\frac{\propto}{384\sqrt{\pi}}}$ **greater than ψ_5**	$N_s = \sqrt{\frac{\propto}{3840\sqrt{\pi}}}$ less than ψ_4
Means the probability density of ψ_4 is greater than ψ_5 more stable	The stability of ψ_5 is less than ψ_4

- **Example:: Make a** comparison between the results of quantum mechanics and classical mechanics for the harmonic Oscillator

Results of classical mech	Results of Q.M
1. The energy $E = \frac{1}{2}mw^2A^2$	The energy $E = \left(n + \frac{1}{2}\right)\hbar w$
2. The minimum energy $E_0 = zero$	The min energy $E_0 = \frac{1}{2}\hbar w$
3. Cannot found in the displacement X greater than A	May be found in the forbidden classical zone
4. Momentum Pc has min- value at $x = 0$ can be take high value at $\|x\| = A$	$momentum$ P_{QM} has maximum value at $x = 0$ and min value at $\|x\| = A$



Figure(5.2) classical and quantum zones