The angular momentum

Analysis of angular momentum

Suppose a particle move at a distance (r) from the center of force with velocity of v' around the axis pass through this point as shown in the figure(6.1).

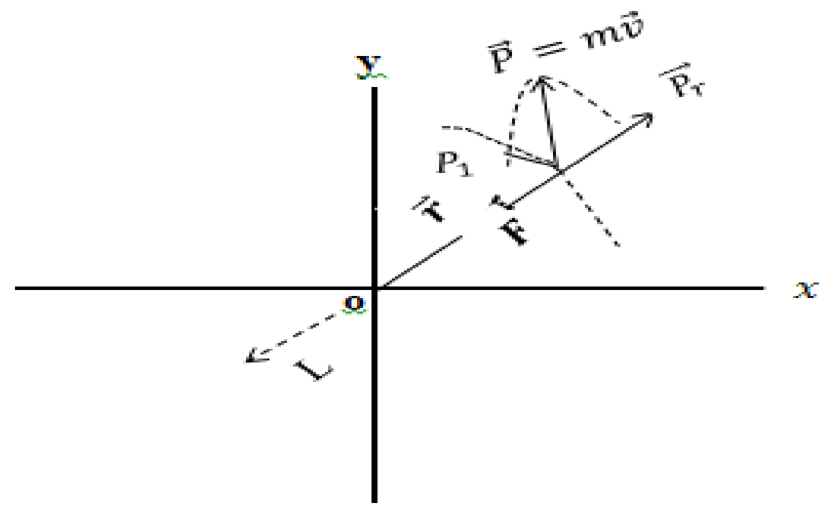

Figure (6.1) Directions of angular momentum

- $\vec{L} = \vec{r} \times P \ldots \ldots . .(6.1)$
- Where P is the Linear momentum and the direction as a tangent of Curvature path of particle.
- Angular momentum in Cartesian coordinates
- By using the quantum formula of linear momentum P
- $\hat{P} = \frac{\hbar}{\dot{\lambda}}\left(\frac{\partial}{\partial x} + \frac{\partial}{dy} + \frac{\partial}{dZ}\right)$ …(6.2)
- $\hat{P} = -\dot{\lambda}\,\hbar\,\nabla \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(6.3)$
- $L = r \times \ (-\dot{\lambda}\,\hbar)\vec{\nabla} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(6.4)$
- $L = -\dot{\lambda}\,\hbar \begin{vmatrix} \dot{\lambda} & J & k \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \ldots \ldots \ldots . \ldots . .(6.5)$

- This means that the components of the angular Momentum as in the following formula.

$$\left.\begin{aligned} \hat{L}_x &= i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ \bullet\ \hat{L}_y &= i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z &= i\hbar\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial x}\right) \end{aligned}\right\} \ldots\ldots\ldots(6.6)$$

- Equation(6.6) represents the quantum formula for angular momentum components

- **Solved problem**: find the commutator of $\left[\hat{L}_x , L_y\right]$
- Solution: $\left[\hat{L}_x , \hat{L}_y\right] = \hat{L}_x \, \hat{L}_y - \hat{L}_y \, \hat{L}_y$
- $\left[L_x , L_y\right]\psi = (i\,\hbar)^2 \, \left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial yz}\right)\psi\right]$
- $= (-\, i\,\hbar)^2 \, \left[y\frac{\partial}{\partial x}\left(z\frac{\partial 4}{\partial y}\right) - y\frac{\partial}{\partial z}\left(x\frac{\partial 4}{\partial z}\right) - z\frac{\partial}{\partial y}\left(z\frac{\partial \psi}{\partial x}\right) + z^2\frac{\partial \psi}{\partial y}\left(x\frac{\partial \psi}{\partial z}\right) - z\frac{\partial}{\partial z}\left(y\frac{\partial \psi}{\partial z}\right) + z\frac{\partial}{\partial x}\left(z\frac{\partial \psi}{\partial y}\right) + x\frac{\partial}{\partial z}\left(y\frac{\partial \psi}{\partial z}\right) - x\frac{\partial}{\partial y}\left(z\frac{\partial \psi}{\partial y}\right)\right.$
- $= (-\, i\,\hbar)^2 \, \left[yz\frac{\partial^2 \psi}{\partial z \partial x} + y\frac{\partial \psi}{\partial x} - \frac{\partial^2 \psi}{\partial z^2} - z^2\frac{\partial^2 \psi}{\partial y \partial x} + zx\frac{\partial^2 \psi}{\partial z \partial x} + zy\frac{\partial^2 \psi}{\partial x \partial z} + z\frac{\partial^2 \psi}{\partial x \partial y} + xy\frac{\partial^2 \psi}{\partial z^2} - xz\frac{\partial^2 \psi}{\partial z \partial x} - x\frac{\partial \psi}{\partial y}\right]$
- $= (-\, i\,\hbar)^2 \, \left[y\frac{\partial \psi}{\partial x} - x\frac{\partial \psi}{\partial y}\right]$
- $= (i\,\hbar)^2 \, \left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right]\psi$
- $\left[L_x , L_y\right]\psi = (i\,\hbar)^2 \, \left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right]\psi$
- $\left[L_x , L_y\right] = (i\,\hbar)(i\,\hbar) \, \left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right]$
- $\therefore \left[L_x , L_y\right] = i\,\hbar \, L_z$ ….(6.7)

- **Use the same method to test your understanding to proving the following**
- $\left[L_y , L_z\right] = i\,\hbar \;\, L_x$ ……(6.8)
- $\left[L_z , L_x\right] = i\,\hbar \;\, L_y$ …….(6.9)
- $\left[L_y , L_x\right] = i\,\hbar \, L_x$ … (6.10)
- $\left[L_z , L_y\right] = -\, i\,\hbar \, L_x$ (6.11)
- $\left[L_x , L_z\right] = -\, i\,\hbar \, L_y$ …….(6.12)
- $L^2 = \dot{L}.\dot{L} = L_x^2 + L_y^2 + L_z^2$ ……(6.13)

- $[A^2, B] = \dot{A}(\hat{A}\hat{B} - \hat{B}\hat{A}) + (\hat{A}\hat{B} - \hat{B}\hat{A})\dot{A} = \hat{A}[\hat{A}, \hat{B}] + [\hat{A}, \hat{B}]\hat{A}$
- **Solved problem**: find $[L^2, L_x]$
- $[L^2, L_x] = L^2 L_x - L_x L^2$
- $= (L_x^2 + L_y^2 + L_z^2)(L_x) - (L_x)(L_x^2 + L_y^2 + L_z^2)$
- $= \underset{(1)}{\underline{L_x^2 L_x}} + \underset{(2)}{\underline{L_y^2 L_x}} + \underset{(3)}{\underline{L_z^2 L_x}} - \underset{(1)}{\underline{L_x L_x^2}} - \underset{(2)}{\underline{L_x L_y^2}} - \underset{(3)}{\underline{L_x L_z^2}}$
- $[L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$
- $\therefore [L_x^2, L_x] = 0$
- By the other method, we can prove $[\hat{L}_y^2, \hat{L}_x] = zero$
- $[\hat{L}_y^2 \hat{L}_x] + [L_z^2, L_x] = L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_y, L_x]$
  $+ [L_y, L_x]L_z$
- and we have $[L_y, L_x] = -\dot{\lambda}\,\hbar\, L_y$
- $[L_z, L_x] = \dot{\lambda}\,\hbar\, L_y$
- $= -\dot{\lambda}\,\hbar\, L_y L_z - \dot{\lambda}\,\hbar\, L_z L_y + \dot{\lambda}\,\hbar L_y L_z + i\hbar L_y L_z$
- $= zero$

- By same procedure we can prove the following
- $\left[L^2 , \hat{L}_x\right] = 0$
- $\left[L^2 , L_y\right] = 0$
- $[L^2 , L_z] = 0$

• The angular momentum in spherical coordinates

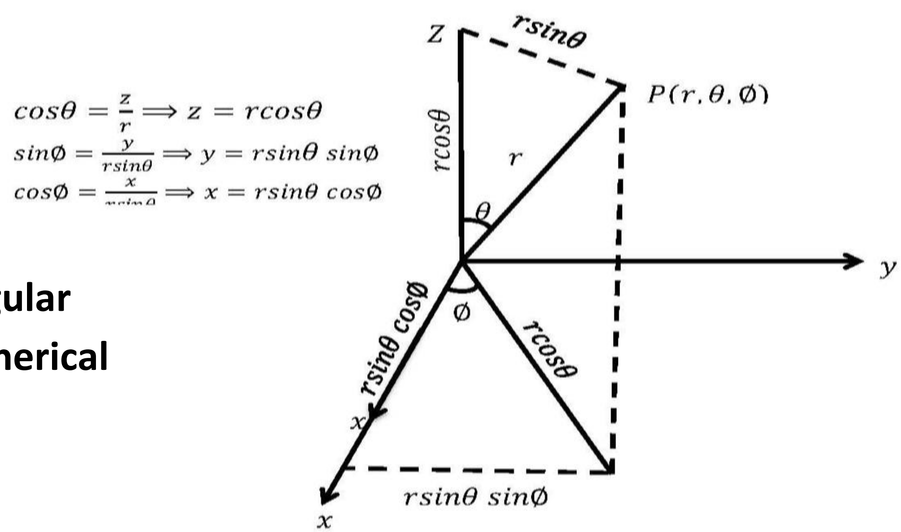

Figure (6.2) angular momentum in spherical coordinates

$$x = r sin\theta\ cos\emptyset$$

- $$y = r sin\theta\ sin\emptyset \ldots\ldots\ldots\ldots..(6.14)$$

$$z = r cos\theta$$

- $$r^2 = x^2 + y^2 + z^2$$
- $$tam\emptyset = \frac{y}{x} =$$
- $$cos\theta = \frac{Z}{r} = \frac{z}{(x^2+y^2+Z^2)^{\frac{1}{2}}} \ldots (6.15)$$
- $$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \Theta}{\partial x}\frac{\partial}{\partial \Theta} + \frac{\partial \emptyset}{\partial x}\frac{\partial}{\partial \emptyset} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \Theta}{\partial y}\frac{\partial}{\partial \Theta} + \frac{\partial \emptyset}{\partial y}\frac{\partial}{\partial \emptyset} \\ \frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial z} + \frac{\partial \Theta}{\partial z}\frac{\partial}{\partial \Theta} + \frac{\partial \emptyset}{\partial z}\frac{\partial}{\partial \emptyset} \end{aligned}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.16)$$

- We have
- $r^2 = x^2 + y^2 + z^2$
- $2r\frac{\partial r}{\partial x} = 2x \quad , 2r\ \frac{\partial r}{\partial y} = 2y \quad , 2r\frac{\partial r}{\partial z} = 2z$
- $$\left.\begin{array}{l} \frac{\partial r}{\partial x} = \frac{2x}{2r} = \frac{x}{r} = \frac{rsin\theta\ cos\emptyset}{r} = sin\theta\ cos\emptyset \\ \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{rsin\theta\ sin\emptyset}{r} = sin\theta\ sin\emptyset \\ \frac{\partial r}{\partial z} = \frac{z}{r} = \frac{rcos\ \emptyset}{r} = cos\emptyset \end{array}\right] \ldots\ldots (6.17)$$
- To $obtain\ \frac{\partial\theta}{\partial x}$

- $Cos\theta = \frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}} \Longrightarrow cos\theta = z(x^2 + y^2 + z^2)^{\frac{1}{2}}$
- $-sin\theta \frac{\partial\theta}{\partial x} = -\frac{1}{2}\ z(x^2 + y^2 + z^2)^{-\frac{3}{2}}(2x) = \frac{-zx}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{-zx}{(r^2)^{\frac{3}{2}}}$
- $-sin\theta \frac{\partial\theta}{\partial x} = \frac{-zx}{r^3} \Longrightarrow\ z\frac{\partial\theta}{\partial x} = \frac{zr}{r^3\ sin\theta}$
- $\therefore \frac{\partial\theta}{\partial x} = \frac{rcos\theta.rsin\theta\ cos\emptyset}{r^3\ sin\theta} = \frac{cos\theta\ cos\emptyset}{r}$
- $\therefore \frac{\partial\theta}{\partial y} = \frac{cos\theta\ cos\emptyset}{r}$
- $\frac{\partial\theta}{\partial y}$
- $Cos\theta = \frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}} = z(x^2 + y^2 + z^2)^{-\frac{1}{2}} \Longrightarrow$
- $-sin\theta \frac{\partial\theta}{dy} = -\frac{1}{2}z(x^2 + y^2 + z^2)^{-\frac{3}{2}}(2y) = \frac{-zy}{r^3}$
- $\frac{\partial\theta}{dy} = \frac{zy}{r^3 sin\theta} = \frac{rcos\theta.rsin\theta\ cos\emptyset}{r^3\ sin\theta}$
- $\boxed{\frac{\partial\theta}{dy} = \frac{cos\theta.sin\emptyset}{r}}$

- **Test your understanding by proving the following**
- $\frac{d\theta}{dz} = -\frac{sin\theta}{r}$
- $tan\emptyset \; \frac{y}{x}$
- $Sec^2\emptyset \; \frac{\partial\emptyset}{\partial x} = -\frac{y}{x^2} = -\frac{rsin\theta . sin\emptyset}{r^2 \; Sin^2\theta \; cos^2\emptyset}$
- $\frac{1}{cos^2\emptyset} \; \frac{\partial\emptyset}{\partial x} = -\frac{sin\theta}{rsin\theta \; cos^2\emptyset}$
- $\boxed{\frac{\partial\emptyset}{\partial x} = \frac{-\, sin\emptyset}{r \; sin\theta}}$ $\boxed{\frac{\partial\emptyset}{\partial y} = \frac{\cos\emptyset}{r \; sin\theta}}$ $\boxed{\frac{\partial\emptyset}{\partial z} = 0}$
- $L_Z = -\,\dot{\lambda}\,\hbar \left(\mathcal{X}\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \right)$ in the Cartesian coordinates

- $$L_Z = -i\hbar \frac{\partial}{\partial \emptyset}$$
  $$L_x = i\hbar \left(sin\emptyset \frac{\partial}{\partial \theta} + \cos\emptyset\ cot\theta \frac{\partial}{\partial \emptyset} \right)$$
  $$L_y = i\hbar \left(sin\emptyset\ cot\theta \frac{\partial}{\partial \theta} - \cos\emptyset \frac{\partial}{\partial \emptyset} \right)$$
  (in the spherical coordinates)....(6.18).
- $L^2 = L_x^2 + L_y^2 + L_z^2$
- $\hat{L}^2 = -\hbar^2 \left[\frac{1}{sin^2\theta} \frac{\partial^2}{\partial \emptyset^2} + \frac{1}{sin\theta} \frac{\partial}{\partial O} \left(sin\theta \frac{\partial}{\partial o} \right) \right]$ .....(6.19)
- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- $\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{\hat{L}^2}{\hbar^2 r^2} \right.$ ....(6.20)