The expectation values:

Let ∝ any quantity then the expectation value of ∝ equals to

$$<\propto>= \int_{all\ space} \psi^{x} \propto_{of} \psi dx......(4.8)$$

Solved problem: Find the expectation value $\langle\propto\rangle$ of a particle in the potential box of one dimension at ground state?

- $< \alpha >= \int_{auspace} \psi^{x} \propto_{op} \psi dx$
- $<\propto>== \int_{o}^{a} \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} \chi \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} dx$
- $= \frac{2}{a} \int_{o}^{a} x \sin^{2} \frac{\pi x}{a} dx \Rightarrow \frac{2}{a} \int_{o}^{a} x \frac{1}{2} \left(1 - \cos \frac{2\pi x}{a}\right) dx$
- $= \frac{1}{a} \left[\int_{o}^{a} x dx - \int_{o}^{a} x \ \cos \frac{2\pi x}{a} dx \right]$
- $= \frac{1}{a} \left\{ \left[\frac{x^{2}}{2} \right]_{o}^{a} - \int_{o}^{a} x \cos \frac{2\pi x}{a} dx \right\}$
- $\frac{1}{a} \left\{ \left[\frac{a^{2}}{2} \right] - \int_{o}^{a} x \cos \frac{2\pi x}{a} dx \right\}$
- Let I $= \int_{o}^{a} x \cos \frac{2\pi x}{a} dx$
- Let $x = u \Rightarrow du = dx$
- $dv = \cos \frac{2\pi x}{a} dx \Rightarrow v == \frac{2\pi}{a} \sin \frac{2\pi x}{a}$

- $\int u dv = uv - \int v du$
- $= x \frac{2\pi}{a} \sin \frac{2\pi x}{a} \Big]_o^a - \int_o^a \frac{a}{2\pi} \sin \frac{2\pi x}{a} dx$
- $= o - \frac{a}{2\pi} \left[\frac{a}{2\pi} \cos \frac{2\pi x}{a} \right]_o^a$
- $\therefore \mathrm{I} = o$
- $\therefore \frac{1}{a} = \left\{ \frac{a^2}{2} - o \right\} = \frac{a}{2}$
- $\therefore \langle x \rangle = \frac{a}{2}$

- **Solved problem:** find the uncertainty or the variance (∇x) in position for particle in P.B. at ground state.
- $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ ……….(4.8)
- We have $\langle x \rangle = \frac{a}{2}$
- $\therefore \langle x \rangle^2 = \frac{a^2}{4}$
- $\langle x^2 \rangle = \int_o^a \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a} x^2 \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a} dx$
- $\langle x^2 \rangle = \frac{2}{a} \int_o^a x^2 \sin^2\frac{\pi x}{a} dx$
- $= \frac{2}{a} \int_o^a x^2 \frac{1}{2}\left(1 - \cos\frac{2\pi x}{a}\right) dx$
- $\frac{1}{a}\left[\int_o^a x^2\, dx - \int_o^a x^2 \cos\frac{2\pi x}{a}\right]$

- The first integral $\int_o^a x^2\,dx = \mathrm{I}_1$
- $\mathrm{I}_1 = \int_o^a x^2\,dx = \left.\frac{x^3}{3}\right]_o^a = \frac{a^3}{3}$
- $\mathrm{I}_2 = \int_o^a x^2 \cos\frac{2\pi x}{a}\,dx$
- $u = x^2 \Rightarrow du = 2x\,dx$
- $dv = \sin\frac{2\pi x}{a} \Rightarrow v = \frac{-a}{2\pi}\cos\frac{2\pi x}{a}$
- $\int u dv = uv - \int v du$
- $\mathrm{I}_2 = \left. x^2 \frac{a}{2\pi}\sin\frac{2\pi x}{a}\right]_o^a - 2\int_o^a x\,\frac{a}{2\pi}\sin\frac{2\pi x}{a}\,dx$
- $\mathrm{I}_2 = o - \frac{a}{\pi}\int_o^a x \sin\frac{2\pi x}{a}\,dx$

- Let $u = x \Rightarrow du = dx$
- $dv = \sin\frac{2\pi x}{a} \Rightarrow v = -\frac{a}{2\pi}\cos\frac{2\pi x}{a}$
- $= \frac{a}{\pi}\left\{\left[\frac{-xa}{2\pi}\cos\frac{2\pi x}{a}\right]_o^a - \int_o^a -\frac{a}{2\pi}\cos\frac{2\pi x}{a}\right\}$
- $= \frac{a}{\pi}\left\{\left(-\frac{a^2}{2\pi}\cos 2\pi\right) + \frac{a}{2\pi}\left[\frac{a}{2\pi}\sin\frac{2\pi x}{a}\right]_o^a\right.$
- $= \frac{a}{\pi}\left[-\frac{a^2}{2\pi}(1)\right] = \frac{a^3}{2\pi^2}$
- The result $= \frac{1}{a}\left[\frac{a^3}{3} - \frac{a^3}{2\pi^2}\right] = \frac{a^2}{3} - \frac{a^2}{2\pi^2}$
- $\therefore \Delta x = \sqrt{\frac{a^2}{3} - \frac{a^2}{2\pi^2} - \frac{a^2}{4}}$ the variance

- **Solved example:** Calculate the momentum rate of particle in P.B. of ground state?
- solution: The momentum rate means $\langle \mathrm{P} \rangle$
- $\langle \mathrm{P} \rangle = \int_o^a \psi_1 \, \mathrm{P}_{\mathrm{o}p} \, \psi_1 \, dx$
- $\int_o^a \sqrt{\frac{2}{a}} \, \sin\frac{\pi x}{a} \, \frac{\hbar}{i} \, \frac{d}{dx} \, \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a} \, dx$
- $= \frac{2\hbar}{ai} \int_o^a \sin\frac{\pi x}{a} \left(\frac{\pi}{a} \cos\frac{\pi x}{a}\right) dx$
- $\therefore \langle \mathrm{P} \rangle = \frac{2\hbar}{ai} \left[\frac{1}{2} \sin^2 \frac{\pi x}{a}\right]_o^a$
- $= \frac{2\hbar}{ai} [zero] = zero.$

- **Solved example:** find the rate of momentum square for particle in P.B. of one dimension in second excited state.
- $\langle P^2 \rangle = \int_o^a \psi_1^* \, P_{op} \psi_3 \, dx$
- $\psi_3 = \sqrt{\frac{2}{a}} \, \sin\frac{3\pi x}{a} = \psi_3^1$
- $\langle P^2 \rangle = \int_o^a \sqrt{\frac{2}{a}} \, \sin\frac{3\pi x}{a} \left(-\hbar^2 \, \frac{\partial 2}{\partial x^2}\right) \sqrt{\frac{2}{a}} \sin\frac{3\pi x}{a} \, dx$
- $\langle P^2 \rangle = \frac{2\hbar^2}{a} \int_o^a \sin\frac{3\pi x}{a} \left(-\frac{\partial}{\partial x}\left(\frac{3\pi}{a}\cos\frac{3\pi x}{a}\right)\right.$
- $= \frac{6\pi\hbar^2}{a^2} \int_o^a \sin\frac{3\pi x}{a}\left(\frac{3\pi}{a}\sin\frac{3\pi x}{a}\right)$
- $= \frac{18\pi^2\hbar^2}{a^3} \int_o^a \sin^2\frac{3\pi x}{a}$
- $= \frac{18\pi^2\hbar^2}{a^3} \int_o^a \frac{1}{2}\left(1 - \cos\frac{3\pi x}{a}\right) dx$

- $= \frac{9\pi^2\hbar^2}{a^3}\left[\int_o^a dx - \int_o^a \cos\frac{3\pi x}{a}\ dx\right]$
- $= \frac{9\pi^2\hbar^2}{a^3}\left\{[a] - \ \left[\frac{a}{3\pi}\sin\frac{3\pi x}{a}\right]_o^a\right.$
- $\langle P^2\rangle = \frac{9\pi^2\hbar^2}{a^2}$
- **** Problems:**
- Find the momentum rate of particle in potential box at first excited state?
- Calculate the momentum rate of particle in potential box at third excited state.

3- Prove the wave function of second excited state is normalized.

4- Prove the wave function of second excited state is orthogonal with the wave function of fourth excited state?

Thank you for your listening