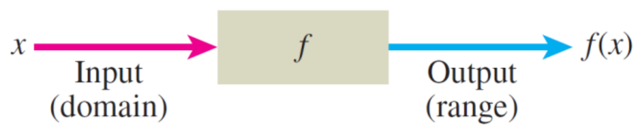
Chapter.1 *Functions*

1.1 Functions and Their Graphs

Functions; Domain and Range

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description.
The set D of all possible input values is called the domain of the function. The set of all values of f(x) as x varies throughout D is called the range.



EXAMPLE. 1: Find the domain and range of each function.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution:

$y = x^2$

Domain $(-\infty, \infty)$

Range $[0, \infty)$

x	$y = x^2$
3	9
2	4
1	1
0	0
-1	1
-2	4
-3	9

$y = \frac{1}{x}$

Domain $(-\infty, 0) \cup (0, \infty)$

Range $(-\infty, 0) \cup (0, \infty)$

x	$y = {}^{1}/_{x}$
3	${}^{1}/_{3}$
2	${}^{1}/_{2}$
1	1
0	${}^{1}/_{0} = error$
-1	-1
-2	$-{}^{1}/_{2}$
-3	$-{}^{1}/_{3}$

$y = \sqrt{x}$

Domain $[0, \infty)$

Range $[0, \infty)$

x	$y = \sqrt{x}$
3	$\sqrt{3}$
2	$\sqrt{2}$
1	$\sqrt{1} = 1$
0	$\sqrt{0} = 0$
-1	$\sqrt{-1} = eroor$
-2	$\sqrt{-2} = eroor$
-3	$\sqrt{-3} = eroor$

$y = \sqrt{4 - x}$

Domain $(-\infty, 4]$

Range $[0, \infty)$

x	$y = \sqrt{4 - x}$
5	$\sqrt{-1} = eroor$
4	$\sqrt{0} = 0$
3	$\sqrt{1}$
2	$\sqrt{2}$
1	$\sqrt{3}$
0	$\sqrt{4}$
-1	$\sqrt{5}$
-2	$\sqrt{6}$

$y = \sqrt{1 - x^2}$

Domain $[-1,1]$

Range $[0,1]$

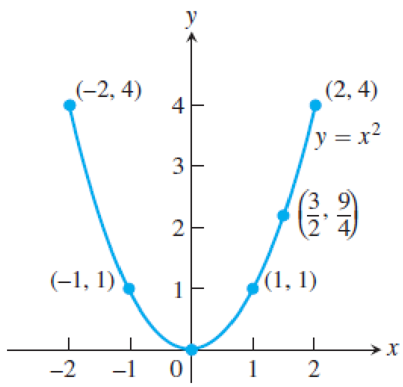
x	$y = \sqrt{1 - x^2}$
3	$\sqrt{-8}$
2	$\sqrt{-3}$
1	$\sqrt{0} = 0$
0	$\sqrt{1} = 1$
-1	$\sqrt{0} = 0$
-2	$\sqrt{-3}$
-3	$\sqrt{-8}$

Graphs of Functions

Ex. 2: Graph the function $y = x^2$ over the interval $[-2,2]$.

Sol.:

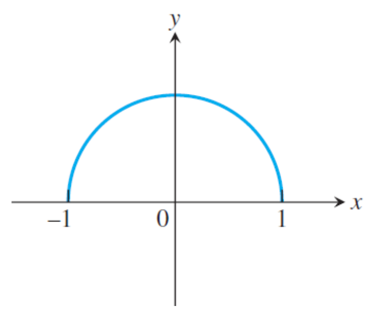
x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



Ex. 3: Graph the function $f(x) = \sqrt{1 - x^2}$.

Sol.:

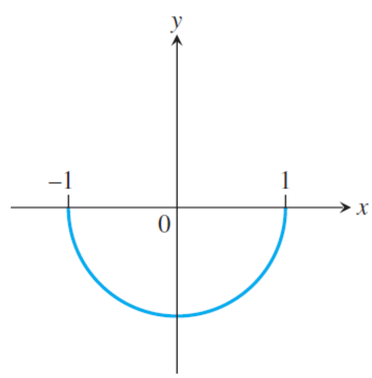
x	$y = \sqrt{1 - x^2}$
2	$\sqrt{-3} = \infty$
1	$\sqrt{0}$ =0
0	1
-1	$\sqrt{0}$ =0
-2	$\sqrt{-3} = \infty$



Ex. 4: Graph the function $f(x) = -\sqrt{1 - x^2}$.

Sol.:

x	$y = -\sqrt{1 - x^2}$
2	$-\sqrt{-3} = \infty$
1	$\sqrt{0}$ =0
0	-1
-1	$\sqrt{0}$ =0
-2	$-\sqrt{-3} = \infty$

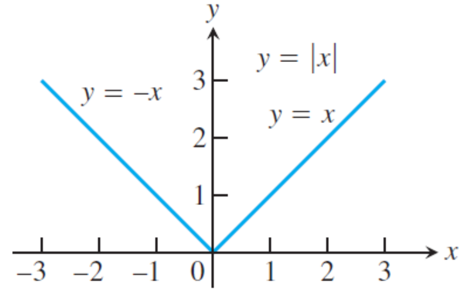


Ex. 5: Graph the function $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

Sol.:

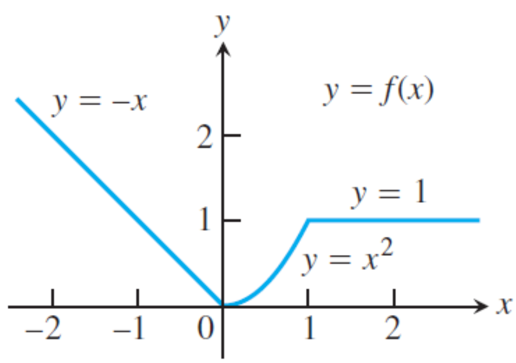
$\lvert x \rvert = x$
3
2
1
0

$\lvert x \rvert = -x$
−3
−2
−1



Ex. 6: Graph the function $f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$

Sol.:

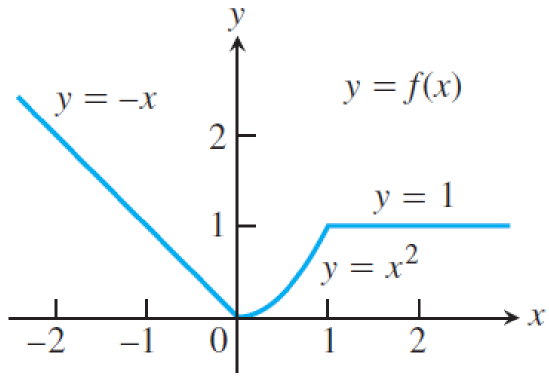


Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is increasing. If the graph descends or falls as you move from left to right, the function is decreasing.

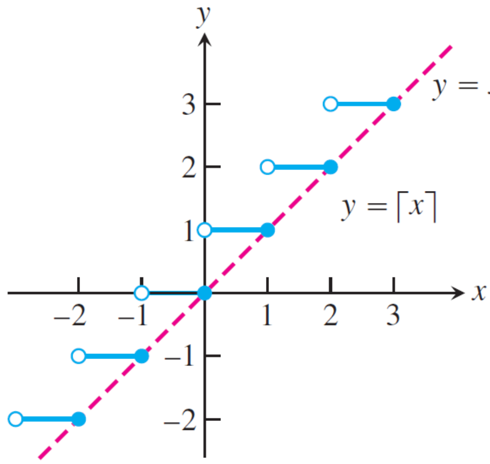
Ex. 7:

The function graphed in Figure is decreasing on $(-\infty, 0]$ and increasing on [0, 1]. The function is neither increasing nor decreasing on the interval $[1, \infty)$.



Ex. 8:

$y = x$



Even Functions and Odd Functions: Symmetry

The graphs of **even** and **odd** functions have characteristic **symmetry** properties. The names even and odd come from **powers** of x. If y is an **even power** of x, as in $y = x^2$ or $y = x^4$ it is an **even function** of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an **odd power** of x, as in $y = x$ or $y = x^3$ it is an **odd function** of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an **even function** is **symmetric about the y-axis**.

The graph of an **odd function** is **symmetric about the origin**.

Note:

The function is even when $f(-x) = f(x)$

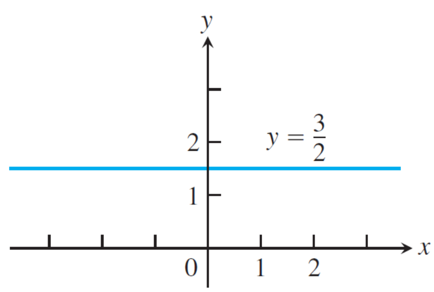
The function is odd when $f(-x) = -f(x)$

Ex. 9:

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.13a).
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b).

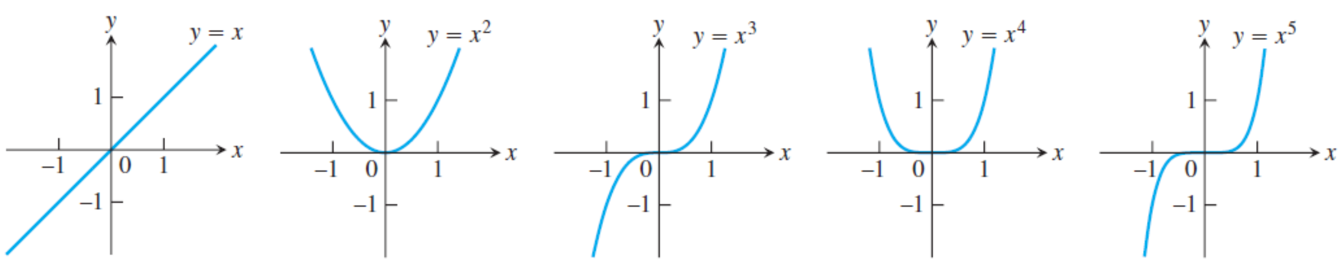
Linear Functions

Ex. 10:



Power Functions

Ex. 11:



H. W.1:

1. Find the domain and range of each function.

a. $f(x) = 1 + x^2$

b. $F(x) = \sqrt{5x + 10}$

c. $f(t) = \frac{4}{3-t}$

2. Find the domain and graph the functions.

a. $f(x) = 5 - 2x$

b. $g(x) = \sqrt{|x|}$

c. $F(t) = t/|t|$

3. Graph the functions.

a. $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2 \end{cases}$

b. $F(x) = \begin{cases} 4 - x^2 & x \le 1 \\ x^2 + 2x & x > 1 \end{cases}$

4. Graph the functions. What symmetries? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

a. $y = -x^3$

b. $y = -\frac{1}{x}$

c. $y = \sqrt{|x|}$

d. $y = x^3/8$

e. $y = -x^{\frac{3}{2}}$

5. In these functions, say whether the function is even, odd, or neither. Give reasons for your answer.

a. $f(x) = 3$

b. $f(x) = x^2 + 1$

c. $g(x) = x^3 + x$

d. $h(t) = 2t + 1$

1.2 Combining Functions, Shifting and Scaling Graphics

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g, we define functions and $f+g$, $f-g$, *and* fg by the formulas

$$(f+g)(x)=f(x)+g(x)$$

$$(f-g)(x)=f(x)-g(x)$$

$$(fg)(x)=f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)} \qquad \text{(where } g(x)\neq 0\text{)}$$

$$(cf)(x)=cf(x)$$

c: constant

Ex.: The functions defined by the formulas

$f(x)=\sqrt{x}$ and $g(x)=\sqrt{1-x}$, find the following

Sol.:

Function	Formula
$f+g$	$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$
$f-g$	$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$
$g-f$	$(g-f)(x)=\sqrt{1-x}-\sqrt{x}$
$f\cdot g$	$(f\cdot g)(x)=f(x)g(x)=\sqrt{x(1-x)}$
f/g	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\sqrt{\frac{x}{1-x}}$
g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$

Composite Functions

Composition is another method for combining functions.

DEFINITION: If f and g are functions, the ***composite*** function $f \circ g$ (f composed with g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

Ex.: If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

a. $(f \circ g)(x)$ b. $(g \circ f)(x)$ c. $(f \circ f)(x)$ d. $(g \circ g)(x)$

Sol.:

(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$

(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$

(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$

(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$

Ex.: If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following:

a. $f(g(0)) = x^2 - 3 + 5 = x^2 + 2 = (0)^2 + 2 = 2$

b. $g(f(0)) = (x + 5)^2 - 3 = x^2 + 10x + 25 - 3 = x^2 + 10x + 22$
$= (0)^2 + 10(0) + 22 = 22$

c. $f(g(x)) = x^2 - 3 + 5 = x^2 + 2$

d. $g(f(x)) = (x + 5)^2 - 3 = x^2 + 10x + 22$

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

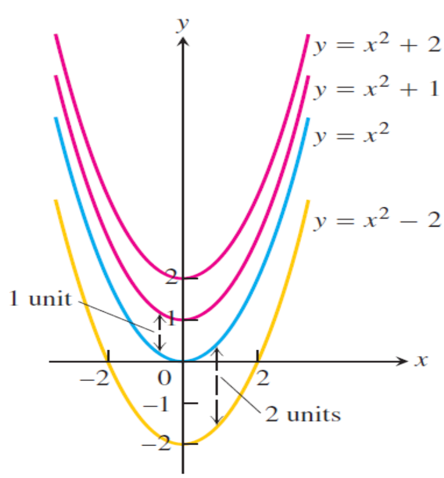
Horizontal Shifts

$y = f(x + h)$ Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

Ex.:

Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit.

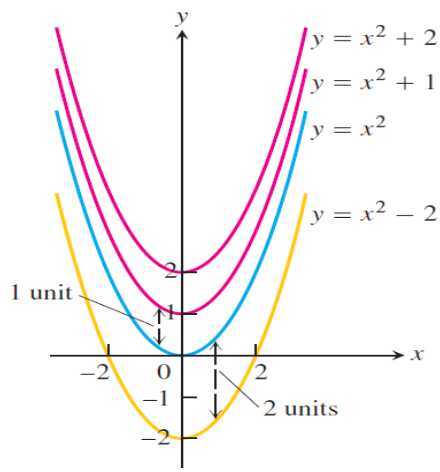
Sol.:



Ex.:

Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ and shifts the graph up 2 units.

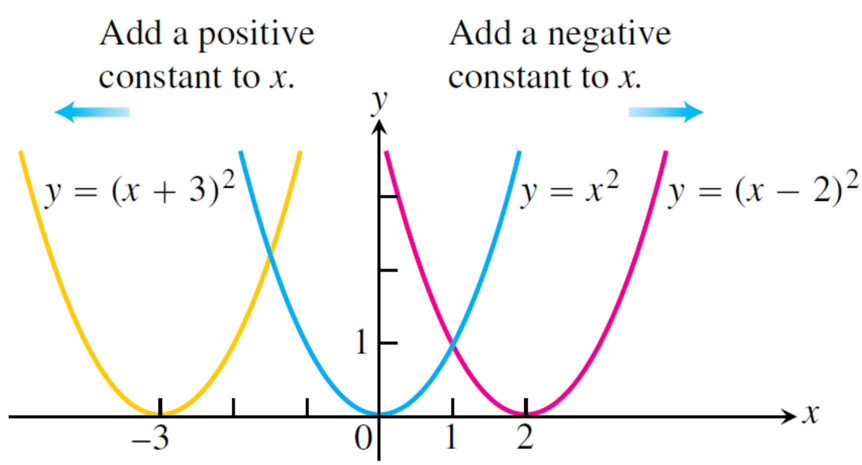
Sol.:



Ex.:

Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left.

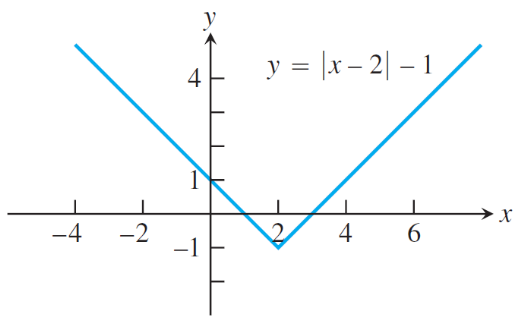
Sol.:



Ex.:

Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x-2| - 1$ and shifts the graph 2 units to the right and 1 unit down.

Sol.:

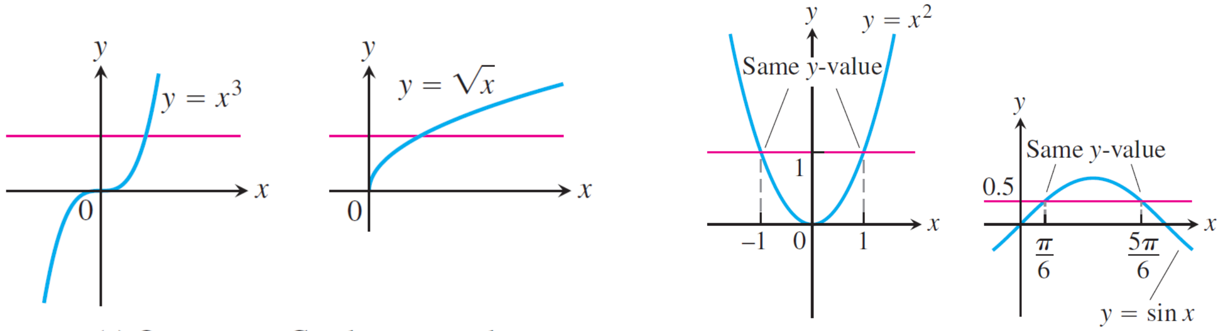


1.6 Inverse Functions and Logarithms

A function that undoes, or inverts, the effect of a function f is called the inverse of f. Many common functions, though not all, are paired with an inverse.

One-to-One Functions

A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$; the sines of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ are both $\frac{\sqrt{3}}{2}$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one. These functions take on any one value in their range exactly once.



(a) One-to-one: Graph meets each horizontal line at most once.

(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

Ex.: Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Sol.:

1. Solve for x in term of y:

$$y = \frac{1}{2}x + 1 \qquad * 2$$

$$2y = x + 2 \qquad \Rightarrow \qquad x = 2y - 2$$

2. Interchange x and y: $\quad y = 2x - 2$

The inverse of the function is the function $f(x) = \frac{1}{2}x + 1$ is the function $f^{-1}(x) = 2x - 2$.

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x$$

Ex.: Find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of x.

Sol.:

1. We first solve for x in term y .

$$y = x^2 \qquad \Rightarrow \qquad x = \sqrt{y}$$

2. Interchange x and y:

$$y = \sqrt{x}$$

The inverse of the function is the function $y = x^2$ is the function $f^{-1}(x) = \sqrt{x}$.

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = \sqrt{x^2} = x$$

$$f(f^{-1}(x)) = \left(\sqrt{x}\right)^2 = x$$

H. W.2:

1. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$ b. $g(f(0))$

c. $f(g(x))$ d. $g(f(x))$

e. $f(f(-5))$ f. $g(g(2))$

g. $f(f(x))$ h. $g(g(x))$

2. Write formulas for $f \circ g \circ h$:

$$f(x) = \sqrt{x+1} \quad , \quad g(x) = \frac{1}{x+4} \quad , \quad h(x) = \frac{1}{x}$$

3. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

$y = x^3$ Left 1, down 1

4. For a function $y = f(x)$, Find a formula for $f^{-1}(x)$ in each case. As a check, show that $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right)$

1. $f(x) = x^2 + 1$ $x \geq 0$
2. $f(x) = x^3 - 1$
3. $f(x) = (x+1)^2$ $x \geq -1$
4. $f(x) = x^5$ $x \geq 0$
5. $f(x) = x^3 + 1$
6. $f(x) = \frac{1}{x^2}$ $x > 0$

Limits

Limits:

It is a study of a function ($y = f(x)$) near a particular point say x_o, but not at x_o.

Properties:

1. $\lim\limits_{x\to a} k = k$, k is constant

Ex.: $\lim\limits_{x\to 3} 5 = 5$

2. $\lim\limits_{x\to a} k\, f(x) = k \lim\limits_{x\to a} f(x) = k\, f(a)$

Ex.: $\lim\limits_{x\to 2} 4x^2 = 4 \lim\limits_{x\to 2} x^2 = 4(2)^2 = 16$

3. $\lim\limits_{x\to a}[f(x) \mp g(x)] = \lim\limits_{x\to a} f(x) \mp \lim\limits_{x\to a} g(x)$

Ex.: $\lim\limits_{x\to 5}(x^2 - 4x + 3) = \lim\limits_{x\to 5} x^2 - \lim\limits_{x\to 5} 4x + \lim\limits_{x\to 5} 3 = 5^2 - 4(5) + 3 = 8$

4. $\lim\limits_{x\to a}[f(x) \cdot g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$

i.e. If $f(x) = g(x)$ then $\lim\limits_{x\to a}[f(x)]^2 = \left(\lim\limits_{x\to a} f(x)\right)^2$

5. $\lim\limits_{x\to a} x^n = \left(\lim\limits_{x\to a} x\right)^n = a^n$

Ex.: $\lim\limits_{x\to 3} x^4 = 3^4 = 81$

6. If $f(x)$ is a rational function $= \frac{P(x)}{g(x)}$ where $P(x)$ & $g(x)$ are polynomials for any real number a

a) If $g(a) \neq 0$ then $\lim\limits_{x\to a} f(x) = f(a)$

b) If $g(a) = 0$ then $\lim\limits_{x\to a} f(x)$ does not exist.

Ex.:

$$\lim_{x\to 4}\frac{2-x}{(x-4)(x+2)}=\frac{2-4}{(4-4)(4+2)}=\frac{-2}{(0)(6)}=-\infty$$

Then the limit does not exist.

7. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Ex.: $\lim_{x\to 2}\frac{5x^3+4}{x-3}=\frac{\lim_{x\to 2} 5x^3+4}{\lim_{x\to 2} x-3}=\frac{5(2)^3+4}{-1}=-44$

General Rule

$$\lim_{x\to a}\sqrt[n]{\mathrm{f(x)}}=\sqrt[n]{\lim_{x\to a} f(x)}=\sqrt[n]{\mathrm{L}}=(L)^{\frac{1}{n}}$$

Ex.: $\lim_{x\to 2}\sqrt[3]{6+\mathrm{x}}=\sqrt[3]{8}=2$

Ex.: Find the limits of the following function as shown below:

1) $\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to 2}(x+2)=4$

2) $\lim_{x\to 5}\frac{x^2-25}{3(x-5)}=\lim_{x\to 5}\frac{(x-5)(x+5)}{3(x-5)}=\lim_{x\to 5}\frac{x+5}{3}=\frac{10}{3}$

3) $\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{(x-1)}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{(x+1)}=\frac{1}{2}$

4) $\lim_{x\to 2}\frac{2x-4}{x^3-2x^2}=\lim_{x\to 2}\frac{2(x-2)}{x^2(x-2)}=\lim_{x\to 2}\frac{2}{x^2}=\frac{2}{4}=\frac{1}{2}$

5) $\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}=\lim_{x\to 0}\frac{x^2(5x+8)}{x^2(3x^2-16)}=\lim_{x\to 0}\frac{5x+8}{3x^2-16}=\frac{-8}{16}=\frac{-1}{2}$

6) $\lim_{x\to c}(x^3+4x^2-3)=\lim_{x\to c}x^3+\lim_{x\to c}4x^2-\lim_{x\to c}3=c^3+4c^2-3$

7) $\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5}=\frac{\lim_{x\to c}(x^4+x^2-1)}{\lim_{x\to c}(x^2+5)}=\frac{\lim_{x\to c}x^4+\lim_{x\to c}x^2-\lim_{x\to c}1}{\lim_{x\to c}x^2+\lim_{x\to c}5}=\frac{c^4+c^2-1}{c^2+5}$

8) $\lim_{x\to 2}\sqrt{4x^2-3}=\sqrt{\lim_{x\to 2}(4x^2-3)}=\sqrt{\lim_{x\to 2}4x^2-\lim_{x\to 2}3}=\sqrt{4(2)^2-3}$

$=\sqrt{16-3}=\sqrt{13}$

9) $\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$

$\frac{x^2+x-2}{x^2-x}=\frac{(x-1)(x+2)}{x(x-1)}=\frac{x+2}{x}$

$$\therefore\quad \lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3$$

Ex.: Evaluate $\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$.

$$\frac{\sqrt{x^2+100}-10}{x^2}=\frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}=\frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$=\frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}=\frac{1}{\left(\sqrt{x^2+100}+10\right)}$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}=\lim_{x\to 0}\frac{1}{\left(\sqrt{x^2+100}+10\right)}=\frac{1}{\left(\sqrt{0^2+100}+10\right)}=\frac{1}{20}=0.05$$

Ex.: Evaluate $\lim_{x\to 2}\frac{\sqrt{x^2+12}-4}{x-2}$.

$$\frac{\sqrt{x^2+12}\ -4}{x-2}=\frac{\sqrt{x^2+12}\ -4}{x-2}\cdot\frac{\sqrt{x^2+12}+4}{\sqrt{x^2+12}+4}$$

$$=\frac{x^2+12\ -16}{(x-2)\left(\sqrt{x^2+12}+4\right)}=\frac{x^2-4}{(x-2)\left(\sqrt{x^2+12}+4\right)}$$

$$=\frac{(x-2)(x+2)}{(x-2)\left(\sqrt{x^2+12}+4\right)}=\frac{(x+2)}{\sqrt{x^2+12}+4}$$

$$\therefore\quad \lim_{x\to 2}\frac{\sqrt{x^2+12}\ -4}{x-2}=\lim_{x\to 2}\frac{(x+2)}{\sqrt{x^2+12}+4}=\frac{2+2}{\sqrt{2^2+12}+4}=\frac{4}{4+4}=\frac{1}{2}$$

Trigonometric Functions:

$\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 0}\frac{x}{\sin x}=1$

$\lim_{x\to 0}\frac{\tan x}{x}=1$

$\lim_{x\to\infty}\sin\frac{1}{x}=0$, $\lim_{\theta\to 0}\sin\theta=0$, $\sin(0)=0$

$\sin^2x+\cos^2x=1$

$\tan x=\frac{\sin x}{\cos x}=\frac{1}{\cot x}$

$\csc^2x-\cot^2x=1$

$1+\cot^2 x=\csc^2x$

$\cot x=\frac{1}{\tan x}=\frac{\cos x}{\sin x}$

$\sec^2x-\tan^2x=1$

$\csc x=\frac{1}{\sin x}$

$1+\tan^2 x=\sec^2x$

$\sec x=\frac{1}{\cos x}$

$\sin^2 3x=\sin 3x.\sin 3x=(\sin 3x)^2$

Ex.:

1) $\lim\limits_{x\to 0}\frac{sin3x}{x} = \lim\limits_{x\to 0}3\frac{sin3x}{3x} = \lim\limits_{x\to 0}3\,.\lim\limits_{x\to 0}\frac{sin3x}{3x} = 3\,.\,1 = 3$

2) $\lim\limits_{x\to 0}\frac{sin2x}{sin3x} = \lim\limits_{x\to 0}\frac{\frac{sin2x}{2x}.2x}{\frac{sin3x}{3x}.3x} = \frac{\lim\limits_{x\to 0}\frac{sin2x}{2x}}{\lim\limits_{x\to 0}\frac{sin3x}{3x}}.\frac{2}{3} = \frac{2}{3}$

Ex.: Find $\lim\limits_{t\to 0}\frac{\tan t\sec 2t}{3t}$

$sec\ 2t = \frac{1}{\cos 2t}$

$$\lim_{t\to 0}\frac{\tan t\sec 2t}{3t} = \frac{1}{3}\lim_{t\to 0}\frac{\sin t}{t}\,.\frac{1}{\cos t}\,.\frac{1}{\cos 2t} = \frac{1}{3}\,.1.1.1 = \frac{1}{3}$$

Ex.: $\lim\limits_{x\to 0}\frac{\cot 4x}{\cot 3x} = \lim\limits_{x\to 0}\frac{\frac{cos4x}{sin4x}}{\frac{cos3x}{sin3x}} = \lim\limits_{x\to 0}\frac{cos4x}{cos3x}.\frac{sin3x}{sin4x}.\frac{3x.4x}{3x.4x} = \frac{3x}{4x} = \frac{3}{4}$

Ex.: $\lim\limits_{x\to 0}\frac{\tan(x-sinx)}{x} = \lim\limits_{x\to 0}\frac{\tan(x-sinx)}{x}.\frac{x-sinx}{x-sinx} = \lim\limits_{x\to 0}\frac{\tan(x-sinx)}{x-sinx}.\lim\limits_{x\to 0}(\frac{x}{x} - \frac{sinx}{x})$

$= 1.(1-1) = 0$

Ex.: $\lim\limits_{x\to 0}\frac{\sin 2x}{5x} = \lim\limits_{x\to 0}\frac{\frac{2}{5}.\ \sin 2x}{\frac{2}{5}.\ 5x} = \frac{2}{5}\lim\limits_{x\to 0}\frac{\sin 2x}{2x} = \frac{2}{5}\,.1 = \frac{2}{5}$

The Sandwich Theorem:

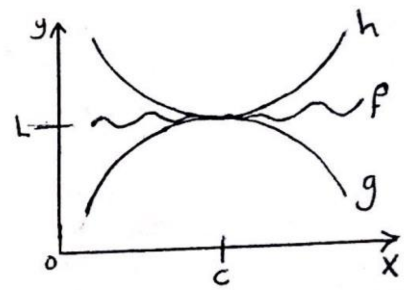
It takes its name because it refers to a function f whose values are sandwich between the values of two other functions g & h that have the same limit L at a point c.

if $$g(x) \leq f(x) \leq h(x)$$

in some open interval containing c expect possibly at $x = c$, and

$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

then $\lim_{x \to c} f(x) = L$



Ex.: if $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$, find $\lim_{x \to 0} u(x)$.

Sol.:

since $\lim_{x \to 0} 1 - \frac{x^2}{4} = 1$ and $\lim_{x \to 0} 1 + \frac{x^2}{2} = 1$

then $\lim_{x \to 0} u(x)=1$

Ex.: if $1 - \frac{x^2}{6} < \frac{x\, sinx}{2-2cosx} < 1$, find the limit of $\frac{x\, sinx}{2-2cosx}$ as $x \to 0$

Sol.:

$\lim_{x \to 0} 1 = 1$, $\lim_{x \to 0} 1 - \frac{x^2}{6} = 1$

then $\lim_{x \to 0} \frac{x\, sinx}{2-2cosx} = 1$

Ex.: if $\frac{1}{2}-\frac{x^2}{24}<\frac{1-cosx}{x^2}<\frac{1}{2}$, find the limit of $\frac{1-cosx}{x^2}$ as $x\to 0$

Sol.:

$\lim_{x\to 0}\frac{1}{2}=\frac{1}{2}$, $\lim_{x\to 0}\frac{1}{2}-\frac{x^2}{24}=\frac{1}{2}$

then $\lim_{x\to 0}\frac{1-cosx}{x^2}=\frac{1}{2}$

Ex.: if $x-\frac{\sin x}{x}+4\le f(x)\le\frac{x^2+2x-3}{x-1}$, find $\lim_{x\to 0}f(x)$ as $x\to 0$

Sol.:

$$\lim_{x\to 0}x-\frac{\sin x}{x}+4=\lim_{x\to 0}x-\lim_{x\to 0}\frac{\sin x}{x}+\lim_{x\to 0}4=0-1+4=3$$

$$\lim_{x\to 0}\frac{x^2+2x-3}{x-1}=\frac{0+0-3}{0-1}=3$$

$\therefore \quad \lim_{x\to 0}f(x)=3$

H.W.3:

1. Find the limit of $\frac{\sqrt{5h+4}-2}{h}$ at $h \to 0$.

2. Find the limits:

1.	$\lim_{x\to -7}(2x+5)$	2.	$\lim_{t\to 6} 8(t-5)(t-7)$
3.	$\lim_{x\to 2}\frac{x+3}{x+6}$	4.	$\lim_{x\to -1} 3(2x-1)^2$
5.	$\lim_{y\to -3}(5-y)^{4/3}$	6.	$\lim_{h\to 0}\frac{3}{\sqrt{3h+1}+1}$
7.	$\lim_{x\to 5}\frac{x-5}{x^2-25}$	8.	$\lim_{x\to -5}\frac{x^2+3x-10}{x+5}$
9.	$\lim_{t\to 1}\frac{t^2+t-2}{t^2-1}$	10.	$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

3. Find the limits:

1.	$\lim_{x\to 0}(2\sin x-1)$	2.	$\lim_{x\to 0}\frac{1+x+\sin x}{3\cos x}$
3.	$\lim_{x\to 0}\frac{\tan 3x}{\sin 8x}$	4.	$\lim_{\theta\to 0}\frac{\sin\sqrt{2\theta}}{\sqrt{2\theta}}$
5.	$\lim_{y\to 0}\frac{\sin 3y}{4y}$	6.	$\lim_{x\to 0}\frac{\tan 2x}{x}$
7.	$\lim_{x\to 0}\frac{x\csc 2x}{\cos 5x}$	8.	$\lim_{x\to 0}\frac{x+x\cos x}{\sin x\cos x}$
9.	$\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin 2\theta}$	10.	$\lim_{t\to 0}\frac{\sin(1-\cos t)}{1-\cos t}$
11.	$\lim_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta}$	12.	$\lim_{\theta\to 0}\frac{\tan\theta}{\theta^2\cot 3\theta}$

4. if $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$, find the limit of $f(x)$ as $x \to 0$

5. if $2-x^2 \le g(x) \le 2\cos x$, find the limit of $g(x)$ as $x \to 0$

6. Prove $\lim_{x\to 0}\frac{\tan x}{x} = 1$

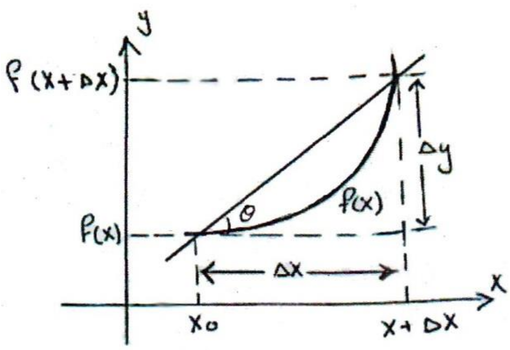
Ch. 3 Differentiation

3.2 The derivative as a Functions

Definition: If $y = f(x)$, then $f'(x) = \frac{dy}{dx} = y' = D_x$ is denoted the derivative is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$f'(x) = y' = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



Ex.: Find $\frac{dy}{dx}$ of $y = x + 5$ by definition.

Sol.:

$y = f(x) = x + 5$, $f(x + \Delta x) = x + \Delta x + 5$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x + 5 - x - 5}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

Ex.: Find $\frac{dy}{dx}$ of $y = x^2 + 5$ by definition.

Sol.:

$y = f(x) = x^2 + 5$, $f(x + \Delta x) = (x + \Delta x)^2 + 5$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 5 - x^2 - 5}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Ex.: Find $\frac{dy}{dx}$ of $y=\sqrt{x}$ by definition.

Sol.:

$y=f(x)=\sqrt{x} \quad , \quad f(x+\Delta x)=\sqrt{x+\Delta x}$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$=\lim_{\Delta x\to 0}\frac{x+\Delta x-x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$
$$(a-b)(a+b)=a^2-b^2$$

تحليل الفرق بين مربعين

Ex.: Find $\frac{dy}{dx}$ of $f(x)=\frac{x}{x-1}$ by definition.

Sol.:

$y=f(x)=\frac{x}{x-1} \quad , \quad f(x+\Delta x)=\frac{x+\Delta x}{x+\Delta x-1}$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{x+\Delta x}{x+\Delta x-1}-\frac{x}{x-1}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{(x-1)(x+\Delta x)-x(x+\Delta x-1)}{(x+\Delta x-1)(x-1)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\frac{(x-1)(x+\Delta x)-x(x+\Delta x-1)}{(x+\Delta x-1)(x-1)}=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\frac{x^2+x\Delta x-x-\Delta x-x^2-x\Delta x+x}{(x+\Delta x-1)(x-1)}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\frac{-\Delta x}{(x+\Delta x-1)(x-1)}=\lim_{\Delta x\to 0}\frac{-1}{(x+\Delta x-1)(x-1)}=\frac{-1}{(x+0-1)(x-1)}$$

$$=\frac{-1}{(x+0-1)(x-1)}=\frac{-1}{(x-1)^2}$$

H. W.: Find $\frac{dy}{dx}$ of x^3+5 by definition.

$$(a+b)^3=a^3+3a^2b+3b^2a+b^3$$
$$(a-b)^3=a^3-3a^2b+3b^2a-b^3$$

فرق مكعبين

3.3 Differentiation Rules

Let c and n are constants, while, v and w are differentiable functions of x.

1. $\frac{d}{dx}c = 0$

Ex.: Find $\frac{dy}{dx}$ for the function $y = 4$

Sol.: $y' = 0$

2. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$ also,

$$\frac{d}{dx}\frac{1}{u^n} = \frac{d}{dx}u^{n-1} = -nu^{-n-1}\frac{du}{dx} = \frac{-n}{u^{n+1}}$$

Ex.: Find $\frac{dy}{dx}$ for the function $y = (x^2+1)^5$.

Sol.: $\frac{dy}{dx} = 5(x^2+1)^4.(2x) = 10x\,(x^2+1)^4$

Ex.: Find $\frac{dy}{dx}$ for the function $y = \frac{1}{x^2}$

Sol.: $y = x^{-2}$ $\rightarrow$ $\frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}$

Ex.: Find $\frac{dy}{dx}$ for the function $y = (2x^3 - 3x^2 + 6x)^{-3}$

Sol.:

$$\frac{dy}{dx} = -3(2x^3 - 3x^2 + 6x)^{-4}.(6x^2 - 6x + 6)$$

$$= -18(2x^3 - 3x^2 + 6x)^{-4}.(x^2 - x + 1)$$

3. $\frac{d}{dx}cu = c\frac{du}{dx}$

Ex.: Find $\frac{dy}{dx}$ for the function $y = 3x^2$

Sol.:

$$\frac{dy}{dx} = 3.2x = 6x$$

4. $\frac{d}{dx}(u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$ and

$$\frac{d}{dx}(u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$$

Ex.: Find $\frac{dy}{dx}$ for the function $y = x^4 + 12x$

Sol.:

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12$$

5. $$\frac{d}{dx}(e^u) = e^u.\frac{du}{dx}$$

Ex.: Find $\frac{dy}{dx}$ for the function $y = e^{2x}$

Sol.:

$$\frac{d}{dx}(e^{2x}) = e^{2x}.\frac{d}{dx}(2x) = e^{2x}.(2) = 2e^{2x}$$

6. $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$ and $\frac{d}{dx}(u.v.w) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

Ex.: Find $\frac{dy}{dx}$ for the function $y = [(5-x)(4-2x)]^2$

Sol.: $\frac{dy}{dx} = 2[(5-x)(4-2x)][(-1)(4-2x) + (-2)(5-x)]$

$= 2[(5-x)(4-2x)][-4 + 2x - 10 + 2x]$

$= 2[(5-x)(4-2x)](-14 + 4x)$

Ex.: Find $\frac{dy}{dx}$ for the function $y = \frac{1}{x}(x^2 + e^x)$

Sol.: $\frac{dy}{dx} = \left(-\frac{1}{x^2}\right)(x^2 + e^x) + (2x + e^x)\frac{1}{x} = 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2}$

$= 1 + \frac{e^x}{x} - \frac{e^x}{x^2}$

7. $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$, where $v \neq 0$

Ex.: Find $\frac{dy}{dx}$ for the function $y = \frac{(x^2+x)(x^2-x+1)}{x^4}$

Sol.: $$y = \frac{x^4-x^3+x^2+x^3-x^2+x}{x^4} = \frac{x^4+x}{x^4} = \frac{x(x^3+1)}{x^4} = \frac{x^3+1}{x^3}$$

$$\frac{dy}{dx} = \frac{x^3.(3x^2)-(x^3+1)(3x^2)}{x^6} = \frac{3x^2[x^3-x^3-1]}{x^6} = \frac{-3}{x^4}$$

Ex.: Find $\frac{dy}{dx}$ for the function $y = \frac{x^2-1}{x^2+x-2}$

Sol.:

$$\frac{dy}{dx} = \frac{(x^2+x-2)(2x)-(x^2-1)(2x+1)}{(x^2+x-2)^2} = \frac{2x^3+2x^2-4x-2x^3-x^2+2x+1}{(x^2+x-2)^2} = \frac{x^2-2x+1}{(x^2+x-2)^2}$$

Ex.: Find $\frac{dy}{dx}$ for the function $y = \frac{2}{x} + \frac{4}{x^3}$

Sol.: $y = 2x^{-1} + 4x^{-3}$

$$\frac{dy}{dx} = -2x^{-2} - 12x^{-4} = \frac{-2}{x^2} - \frac{12}{x^4}$$

Ex.: Find $\frac{dy}{dx}$ for the function $y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$

Sol.:

$$y = 12x^{-1} - 4x^{-3} + 3x^{-4}$$

$$\frac{dy}{dx} = -12x^{-2} + 12x^{-4} - 12x^{-5} = \frac{-12}{x^2} + \frac{12}{x^4} - \frac{12}{x^5}$$

H. W.: Differentiate the following powers of x.

1. x^3
2. $x^{2/3}$
3. $x^{\sqrt{2}}$
4. $\frac{1}{x^4}$
5. $x^{-4/3}$
6. $y = \frac{t^2-1}{t^3+1}$
7. $y = x^3 + \frac{4}{3}x^2 - 5x + 1$
8. $y = x^4 - 2x^2 + 2$
9. $y = e^{-x}$
10. $y = \frac{(x-1)(x^2-2x)}{x^4}$

Higher Derivatives

$f'(x) = \frac{dy}{dx} = y'$ 1st Derivative

$f''(x) = \frac{d^2y}{dx^2} = y''$ 2nd Derivative

$f'''(x) = \frac{d^3y}{dx^3} = y'''$ 3rd Derivative

$f^n(x) = \frac{d^ny}{dx^n} = y^n$ n-th Derivative

Ex.: Find the third derivatives of the following function

$$y = \frac{1}{x} + \sqrt{x^3}$$

Sol.:

$$\frac{dy}{dx} = \frac{-1}{x^2} + \frac{1}{2}(x^3)^{-\frac{1}{2}}(3x^2) = \frac{-1}{x^2} + \frac{3}{2}\frac{x^2}{\sqrt{x^3}} = \frac{-1}{x^2} + \frac{3}{2}\sqrt{x} = \frac{-1}{x^2} + \frac{3}{2}(x)^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{-(-2x)}{x^4} + \frac{3}{4}(x)^{-\frac{1}{2}} = \frac{2}{x^3} + \frac{3}{4\sqrt{x}}$$

$$\frac{d^3y}{dx^3} = \frac{-6}{x^4} - \frac{3}{8}(x)^{\frac{-3}{2}} = \frac{-6}{x^4} - \frac{3}{8\sqrt{x^3}}$$

H. W.: Find the first and second derivatives.

1. $y = -x^2 + 3$

2. $y = \frac{4x^3}{3} - x + 2e^x$

3. $y = 6x^2 - 10x - 5x^{-2}$

4. $y = \frac{x^3+7}{x}$

5. If $y = x \sin x$ find y'''.

3.6 The Chain Rule

Suppose that $y = g(t)$ and $x = f(t)$ then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Ex.: Use the chain rule and find $\frac{dy}{dx}$ in terms of x and y.

1. $y = \frac{t^2}{t^2+1}$ and $t = \sqrt{2x+1}$

Sol.:

$$\frac{dy}{dt} = \frac{(t^2+1)(2t) - t^2(2t)}{(t^2+1)^2} = \frac{2t(t^2+1-t^2)}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$$

$$\frac{dt}{dx} = \frac{1}{2}(2x+1)^{\frac{-1}{2}}.(2) = \frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

or $= \frac{2}{(2x+2)^2} = \frac{2}{4x^2+8x+4} = \frac{2}{4(x^2+2x+1)} = \frac{1}{2(x^2+2x+1)} = \frac{1}{2(x+1)^2}$

2. $y = \frac{1}{t^2+1}$ and $x = \sqrt{4t-1}$

Sol.:

$x^2 = 4t - 1 \quad \rightarrow \quad 4t = x^2 + 1 \quad \rightarrow \quad t = \frac{x^2+1}{4}$

$$\frac{dy}{dt} = \frac{(t^2+1)(0) - (2t)}{(t^2+1)^2} = \frac{-2t}{(t^2+1)^2}$$

$$\frac{dx}{dt} = \frac{1}{2}(4t-1)^{-\frac{1}{2}}.4 = \frac{4}{2}\frac{1}{\sqrt{4t-1}} = \frac{2}{\sqrt{4t-1}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{-2t}{(t^2+1)^2}.\frac{\sqrt{4t-1}}{2} = -y^2x.\frac{(x^2+1)}{4} = \frac{-xy^2(x^2+1)}{4}$$

3. $y = \frac{(t-1)^2}{(t+1)^2}$ and $x = \frac{1}{t^2} - 1$ at $t = 2$

Sol.:

$$\frac{dy}{dt} = \frac{(t^2+1).2.(t-1)(1) - (t^2-1).2.(t+1).(1)}{(t+1)^4}$$

$$= \frac{2(t+1)(t-1)[t+1-t+1]}{(t+1)^4} = \frac{4(t-1)}{(t+1)^3} \quad \rightarrow \quad \left.\frac{dy}{dt}\right|_{t=2} = \frac{4(2-1)}{(2+1)^3} = \frac{4}{27}$$

$$\frac{dx}{dt} = \frac{-2}{t^3} \quad \rightarrow \quad \left.\frac{dx}{dt}\right|_{t=2} = \frac{-2}{(2)^3} = \frac{-2}{8} = \frac{-1}{4}$$

$$\frac{dy}{dx} = \frac{\left.\frac{dy}{dt}\right|_{t=2}}{\left.\frac{dx}{dt}\right|_{t=2}} = \frac{\frac{4}{27}}{\frac{-1}{4}} = \frac{-16}{27}$$

4. $y = 1 - \frac{1}{t}$ and $t = \frac{1}{1-x}$ at $x = 2$

Sol.:

$$\frac{dy}{dt} = \frac{1}{t^2} \quad , \quad \frac{dt}{dx} = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{1}{t^2}.\frac{1}{(1-x)^2} = \frac{1}{\left(\frac{1}{1-x}\right)^2}.\frac{1}{(1-x)^2}$$

$$\frac{dy}{dx} = \left.\frac{dy}{dt}\right|_{x=2}.\left.\frac{dt}{dx}\right|_{x=2} = 1.1 = 1$$

H. W.: Use the chain rule and find $\frac{dy}{dx}$ in terms of x and y.

1. $y = 6t - 9$ and $t = \frac{1}{2}x^4$
2. $y = \sin t$ and $t = 3x + 1$
3. $y = \left(\frac{u}{5}\right) + 7$ and $u = 5x - 35$
4. $y = 1 + \left(\frac{1}{u}\right)$ and $u = \frac{1}{(x-1)}$

3.7 Implicit Differentiation

If the formula for f is an algebraic combination of power of x and y. To calculate the derivatives of these implicitly defined function, we simply differentiate both sides of the defining equation with respect to x. Where $y' = \frac{dy}{dx}$.

Ex.: Find $\frac{dy}{dx}$ for the following functions:

a. $x^2y^2 = x^2 + y^2$

Sol.:

$$2xy^2 + 2y\, y'x^2 = 2x + 2y\, y'$$

$$2x^2y\, y' - 2y\, y' = 2x - 2xy^2 \quad \Rightarrow \quad 2y'(x^2y - y) = 2(x - xy^2)$$

$$y' = \frac{x - xy^2}{x^2y - y}$$

b. $(x + y)^3 + (x - y)^3 = x^4 + y^4$

Sol.:

$$3(x + y)^2(1 + y') + 3(x - y)^2(1 - y') = 4x^3 + 4y^3\, y'$$

$$3(x + y)^2 + 3y'(x + y)^2 + 3(x - y)^2 - 3y'(x - y)^2 = 4x^3 + 4y^3\, y'$$

$$3y'(x + y)^2 - 3y'(x - y)^2 - 4y^3\, y' = 4x^3 - 3(x + y)^2 - 3(x - y)^2$$

$$y' = \frac{4x^3 - 3(x + y)^2 - 3(x - y)^2}{3(x + y)^2 - 3(x - y)^2 - 4y^3} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

c. $\frac{x-y}{x-2y}=2$ at $P(3,1)$

Sol.:

$$\frac{(x-2y)(1-y')-(x-y)(1-2y')}{(x-2y)^2}=0$$

$$(x-2y)-y'(x-2y)-(x-y)+2y'(x-y)=0$$

$$y'[2(x-y)-(x-2y)]=(x-y)-(x-2y)$$

$$y'=\frac{x-y-x+2y}{2x-2y-x+2y}=\frac{y}{x}$$

$$y'|_{(3,1)}=\frac{y}{x}=\frac{1}{3}$$

d. $xy+2x-5y=2$ at $P(3,2)$

Sol.:

$$y+xy'+2-5y'=0$$

$$y'(5-x)=y+2$$

$$y'=\frac{y+2}{5-x}$$

$$y'|_{(3,2)}=\frac{2+2}{5-3}=\frac{4}{2}=2$$

H. W.: Use implicit differentiation to find $\frac{dy}{dx}$.

1. $x^2y+xy^2=6$
2. $2xy+y^2=x+y$
3. $x^2(x-y)^2=x^2-y^2$

Trigonometric Functions

If u is any differentiable function of x then

1. $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$
2. $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$
3. $\frac{d}{dx}\tan u = sec^2 u \frac{du}{dx}$
4. $\frac{d}{dx}\cot u = -csc^2 u \frac{du}{dx}$
5. $\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$
6. $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$

Ex.: Find $\frac{dy}{dx}$ for the following function $y = \tan(3x^2)$

Sol.:

$$\frac{dy}{dx} = sec^2(3x^2).6x = 6x\ sec^2(3x^2)$$

Ex.: Find $\frac{dy}{dx}$ for the following function $y = \tan x \sin x$

Sol.:

$$y = \tan x \sin x = \frac{\sin x}{\cos x} \cdot \sin x = \frac{sin^2 x}{\cos x}$$

$$\frac{dy}{dx} = \frac{2\ cos\ x \sin x \cos x + sin^2 x \sin x}{cos^2 x}$$

$$= \frac{2\ cos^2 x \sin x + sin^2 x \sin x}{cos^2 x} = \frac{2\ cos^2 x \sin x}{cos^2 x} + \frac{sin^2 x \sin x}{cos^2 x}$$

$$= 2\sin x + \sin x \frac{sin^2 x}{cos^2 x} = \sin x\,(2 + tan^2 x)$$

Ex.: Find $\frac{dy}{dx}$ for the following function $y = 2\sin\frac{x}{2} - x\cos\frac{x}{2}$

Sol.:

$$\frac{dy}{dx} = 2\,cos\,\frac{x}{2}\cdot\frac{1}{2} - \left[\cos\frac{x}{2} - x sin\,\frac{x}{2}\cdot\frac{1}{2}\right]$$

$$= \cos\frac{x}{2} - \cos\frac{x}{2} + \frac{x}{2}\sin\frac{x}{2} = \frac{x}{2}\sin\frac{x}{2}$$

Ex.: Prove that: $\frac{d}{dx}(\tan u) = sec^2 u\frac{du}{dx}$.

Sol.:

$$\tan u = \frac{\sin u}{\cos u}$$

$$\frac{d}{dx}\tan u = \frac{\cos u.\cos u\frac{du}{dx} + \sin u.\sin u\,\frac{du}{dx}}{cos^2 u}$$

$$= \frac{\frac{du}{dx}\left[cos^2 u + sin^2 u\right]}{cos^2 u}$$

Since, $cos^2 u + sin^2 u = 1$ and $\sec u = \frac{1}{\cos u}$

$$\therefore \quad \frac{d}{dx}\tan u = sec^2 u\frac{du}{dx}$$

H. W.:

1. Prove that:

a). $\frac{d}{dx}(\sec u) = sec\, u. \tan u \frac{du}{dx}$

b). $\frac{d}{dx}(\csc u) = -csc\, u. \cot u \frac{du}{dx}$

c). $\frac{d}{dx} \sec x = \tan x \sec x$

d). $\frac{d}{dx} \csc x = -\csc x. cot\, x$

e). $\frac{d}{dx}\cot x = -\csc^2 x$

f). $\frac{d}{dx}\sin x = \cos x$

g). $\frac{d}{dx}\cos x = -\sin x$.

2. Use implicit differentiation to find $\frac{dy}{dx}$.

a). $y = sin\, x^3$

b). $y = \sqrt{\sec(3x^2)}$

c). $y = \cos(x^3 + 1)$

d). $y = \frac{1}{2} + \frac{1}{2} cos^2 2x$

e). $y = \frac{\cos x}{x}$

f). $y = x^2 \sin x$

g). $y = 2\sin^2 3x$

h). $x + \tan(xy) = 1$

i). $x^2 = siny + sin2y$

j). $e^{2x} = \sin(x + 3y)$

The inverse trigonometric functions

If u is any differentiable function of x , then:

1) $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, $-1 < u < 1$

2) $\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$, $-1 < u < 1$

3) $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

4) $\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2}\frac{du}{dx}$

5) $\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$, $|u| > 1$

6) $\frac{d}{dx}\csc^{-1}u = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$, $|u| > 1$

Ex.: Find $\frac{dy}{dx}$ in each of the following functions:

a). $y = \cot^{-1}\frac{2}{x} + \tan^{-1}\frac{x}{2}$

Sol.:

$$\frac{dy}{dx} = \left(\frac{-1}{1+\left(\frac{2}{x}\right)^2}\right)\left(\frac{-2}{x^2}\right) + \frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}$$

$$= \frac{-1}{1+\frac{4}{x^2}}\cdot\frac{-2}{x^2} + \frac{1}{1+\frac{x^2}{4}}\cdot\frac{1}{2} = \frac{2}{x^2\left(\frac{x^2+4}{x^2}\right)} + \frac{1}{2\left(\frac{4+x^2}{4}\right)}$$

$$= \frac{2}{x^2+4} + \frac{1}{\frac{x^2+4}{2}} = \frac{2}{x^2+4} + \frac{2}{x^2+4} = \frac{4}{x^2+4}$$

b). $y = \sin^{-1}\frac{x-1}{x+1}$

Sol.:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(x+1).1-(x-1).1}{(x+1)^2}$$

$$= \frac{1}{\sqrt{\frac{(x+1)^2-(x-1)^2}{(x+1)^2}}} \cdot \frac{x+1-x+1}{(x+1)^2}$$

$$= \frac{(x+1)}{\sqrt{x^2+2x+1-x^2+2x-1}} \cdot \frac{2}{(x+1)^2} = \frac{1}{\sqrt{4x}} \cdot \frac{2}{(x+1)} = \frac{1}{\sqrt{x}(x+1)}$$

c). $y = x\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2}$

Sol.:

$$\frac{dy}{dx} = x\frac{(-1)}{\sqrt{1-4x^2}} \cdot 2 + \cos^{-1}2x - \frac{1}{4} \cdot \frac{(-8x)}{\sqrt{1-4x^2}}$$

$$= \frac{-2x}{\sqrt{1-4x^2}} + \cos^{-1}2x + \frac{2x}{\sqrt{1-4x^2}} = \cos^{-1}2x$$

d). $y = \sec^{-1}6x$

Sol.:

$$\frac{dy}{dx} = \frac{6}{|6x|\sqrt{36x^2-1}} = \frac{1}{|x|\sqrt{36x^2-1}}$$

Examples: Prove that:

a) $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

Sol.:

$y = \sin^{-1}u$, $\sin y = u$, $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$

$\cos y\,\frac{dy}{dx} = \frac{du}{dx}$, $\cos^2 x = 1 - \sin^2 x$

$$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx} = \frac{1}{\mp\sqrt{1-\sin^2 y}}\frac{du}{dx} = \frac{1}{\mp\sqrt{1-u^2}}\frac{du}{dx}$$

Since $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y$ is not negative.

Hence, $\cos y = \sqrt{1-u^2}$ $\therefore$ $\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

b) $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

Sol.:

$y = \tan^{-1}u \quad , \quad tan\ y = u \quad , \quad \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

$\frac{d}{dx}\tan y = \frac{du}{dx}$

$\sec^2 y\,\frac{dy}{dx} = \frac{du}{dx} \quad , \quad \sec^2 x - \tan^2 x = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}\frac{du}{dx} = \frac{1}{1+\tan^2 y}\frac{du}{dx} \quad \therefore \quad \frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

c) $\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

Sol.:

$y = \sec^{-1}u \quad , \quad 0 \leq y \leq \pi$

$u = sec\ y$

$\frac{du}{dx} = sec y\tan y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{sec y\tan y}\frac{du}{dx} \quad , \quad$ where $\quad \tan y = \mp\sqrt{\sec^2 y - 1} = \mp\sqrt{u^2 - 1}$

$$\frac{dy}{dx} = \frac{1}{u\left(\mp\sqrt{u^2-1}\right)}\frac{du}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

$1 + \tan^2 x = \sec^2 x \quad \Rightarrow \quad \tan^2 x = \sec^2 x - 1$

H. W.:

1. Find $\frac{dy}{dx}$ in each of the following functions:

a. $sin\left[\cos^{-1}(\frac{\sqrt{2}}{2})\right]$

b. $tan\left[\sin^{-1}(\frac{-1}{2})\right]$

c. $csc[\sec^{-1}(2)] + cos\left[\tan^{-1}(-\sqrt{3})\right]$

2. Prove $\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$.

3. Prove $\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$.

4. Prove $\frac{d}{dx}\csc^{-1}u = \frac{-1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx}$.

Exponential function:

If a is a positive number and x is any number, we define the exponential function as:

$$y = a^x$$

With Domain $-\infty < x < \infty$, Range $y > 0$

The properties of the exponential are:

1) If $a \geq 0 \Rightarrow a^x \geq 0$
2) $a^x \cdot a^y = a^{x+y}$
3) $\frac{a^x}{a^y} = a^{x-y}$
4) $(a^x)^y = a^{x \cdot y}$
5) $(a \cdot b)^x = a^x \cdot b^x$
6) $\sqrt[y]{a^x} = a^{\frac{x}{y}} = \left(\sqrt[y]{a}\right)^x$
7) $a^{-x} = \frac{1}{a^x}$ and $a^x = \frac{1}{a^{-x}}$
8) $a^x = a^y \Rightarrow x = y$
9) $a^0 = 1$, $a^1 = a$, $a^\infty = \infty$, $a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0$

Examples:

1. $x^2 \cdot x^5 = x^7$
2. $\frac{x^2}{x^5} = x^{2-5} = x^{-3}$
3. $(x^3)^2 = x^6$
4. $(xy)^3 = x^3y^3$
5. $x^{\frac{2}{3}} = \sqrt[3]{x^2} = \left(\sqrt[3]{x}\right)^2$
6. $x^{-3} = \frac{1}{x^3}$ and $x^3 = \frac{1}{x^{-3}}$
7. $x^0 = 1$ and $x^1 = x$
8. $x^\infty = \infty$
9. $x^{-\infty} = 0$

Logarithms function:

The definition of the logarithm function, if $a > 0$ & $a \neq 1$ and $x > 0$ then,

$$y = \log_a x \qquad \text{is equivalent to} \qquad a^y = x$$

In this definition $y = \log_a x$ is called *logarithm form* and $a^y = x$ is called *exponential form*.

The properties of the logarithms, if $a > 0$ & $a \neq 1$, $b > 0,$ $c > 0$ *and* n any real, then:

1) $\log_a(b.c) = \log_a b + \log_a c$
2) $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$
3) $\log_a(b^n) = n\log_a b$

Ex.:

$$\log_{10} 100 = \log_{10} 10^2 = 2\log_{10} 10 = 2$$

4) $\log_a\left(\frac{1}{c}\right) = -\log_a c$

Ex.:

$$\log_{10}\frac{1}{1000} = \log_{10} 10^{-3} = -3\log_{10} 10 = -3$$

5) $\log_{10}(u \mp v) \neq \log_a u \mp \log_a v$
6) $\log_a a = 1$
7) $\log_a 1 = 0$

The natural logarithm function is defined by

$$y = \log_e x = \ln x$$

Where $y = \ln x$ is equivalent to $x = e^y$. Let $x, y > 0$ then the properties of natural logarithm functions are:

1) $y = a^x \Rightarrow x = \log_a y \quad and \quad y = e^x \quad \Rightarrow \quad x = \ln y$
2) $\log_e x = \ln x$
3) $\log_a x = \frac{\ln x}{\ln a}$
4) $\ln xy = \ln x + \ln y$
5) $\ln \frac{x}{y} = \ln x - \ln y$
6) $\ln x^n = n \ln x$
7) $e^{x\, lna} = a^x$
8) $e^{lnx} = x \quad , \quad e^{-lnx} = \frac{1}{x} \quad , \quad a^{\log_a x} = x$
9) $ln\, e = 1$ and $ln\, 1 = 0$

$$ln \frac{1}{e} = ln\, e^{-1} = -1$$

Derivatives of Exponential Functions:

1. $\frac{d}{du} a^u = a^u \ln a \frac{du}{dx}$

2. $\frac{d}{du} e^u = e^u \frac{du}{dx}$

3. $\frac{d}{du} \ln u = \frac{1}{u} \frac{du}{dx}$

Ex.: Find $\frac{dy}{dx}$ for the following functions:

a). $y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x}(\ln 2).3$

b). $y = 2^x.3^x \Rightarrow \frac{dy}{dx} = 6^x ln6.1$ $(a.b)^x = a^x.b^x$

c). $y = (2^x)^2 \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x}(ln2).2^1 = 2^{2x+1}.ln2$

d). $y = x.2^{x^2} \Rightarrow \frac{dy}{dx} = x.2^{x^2} ln2(2x) + 2^{x^2}$

e). $y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{x+e^{5x}}.(1 + 5e^{5x})$

f). $y = e^{\sqrt{1+5x^2}} \Rightarrow y = e^{(1+5x^2)^{\frac{1}{2}}}$

$$\frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}}.\left(\frac{1}{2}\right)(1 + 5x^2)^{\frac{-1}{2}}.(10x)$$

$$= e^{(1+5x^2)^{\frac{1}{2}}}.\frac{5x}{\sqrt{1 + 5x^2}}$$

Notes:

$\log_a x = \frac{\ln x}{\ln a}$, $ln\, e = 1$, $\frac{d}{dx} ln\, 10 = 0$

Ex.: Find $\frac{dy}{dx}$ for the following functions:

1). $y = \log_{10} e^x$

Sol.:

$y = x \log_{10} e \Rightarrow y = x\frac{\ln e}{\ln 10} = \frac{x}{\ln 10} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 10}$

2). $y = \log_5 (x+1)^2$

Sol.:

$$y = 2\log_5(x+1) = 2\frac{\ln(x+1)}{\ln 5} = \frac{2}{\ln 5}\ln(x+1)$$

$$\frac{dy}{dx} = \frac{2}{\ln 5}\frac{1}{(x+1)} = \frac{2}{(x+1)\ln 5}$$

3). $y = \log_2(3x^2+1)^3$

Sol.:

$$y = 3\log_2(3x^2+1) = 3\frac{\ln(3x^2+1)}{\ln 2}$$

$$\frac{dy}{dx} = \frac{3}{\ln 2}\frac{6x}{(3x^2+1)} = \frac{18x}{(3x^2+1)\ln 2}$$

4). $y = [\ln(x^2+2)^2]^3$

Sol.:

$$\frac{dy}{dx} = 3[\,2\ln(x^2+2)]^2\left(\frac{4x}{x^2+2}\right) = \frac{12x[\,2\ln(x^2+2)]^2}{(x^2+2)}$$

5). $y + \ln xy = 1$

Sol.:

$$y + \ln x + \ln y = 1 \qquad \Rightarrow \qquad \frac{dy}{dx} + \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(1+\frac{1}{y}\right) = \frac{-1}{x} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{-y}{x(y+1)}$$

H. W.:

Find the derivative of y with respect to x, t or as appropriate:

1) $y = e^{5-7x}$

2) $y = \ln 3t\, e^{-t}$

3) $y = xe^x - e^x$

4) $y = (x^2-2x+2)e^x$

5) $y = e^{\theta}(\sin\theta + \cos\theta)$

6) $y = \ln\frac{e^{\theta}}{1+e^{\theta}}$

7) $y = e^{(\cos t + \ln t)}$

8) $\ln y = e^y \sin x$

Hyperbolic Function:

1. $\sinh x = \frac{e^x - e^{-x}}{2}$
2. $\cosh x = \frac{e^x + e^{-x}}{2}$
3. $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\text{sech}\, x = \frac{2}{e^x + e^{-x}}$
6. $\text{csch}\, x = \frac{2}{e^x - e^{-x}}$

Ex.: Simplify:

1). $\cosh 5x + \sinh 5x$

Sol.:

$$\cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = \frac{2e^{5x}}{2} = e^{5x}$$

2). $2 \cosh \ln x$

Sol.:

$$2 \cosh \ln x = 2\left(\frac{e^{lnx} + e^{-lnx}}{2}\right) = e^{lnx} + e^{lnx^{-1}} = x + x^{-1}$$

Note: $e^{lnx} = x$

3). $(\sinh \mathrm{x} + \cosh x)^4 =$

Sol.:

$$\left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4 = \left(\frac{e^x - e^{-x} + e^x + e^{-x}}{2}\right)^4 = (e^x)^4 = e^{4x}$$

4). $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

Sol.:

$$= \ln\left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right) + \ln\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)$$

$$= \ln\left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right) + \ln\left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right)$$

$$= ln\frac{2e^x}{2} + ln\frac{2e^{-x}}{2} = lne^x + lne^{-x} = x - x = 0$$

Derivative of Hyperbolic Functions:

If u is any differentiable function of x then:

1. $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$
2. $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$
3. $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$
4. $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx}$
5. $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$
6. $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

Hyperbolic Identities:

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \operatorname{sech}^2 x = 1$
3. $\coth^2 x - \operatorname{csch}^2 x = 1$
4. $\sinh 2x = 2\sinh x \cosh x$
5. $\cosh 2x = \cosh^2 x + \sinh^2 x$
6. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$
7. $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$
8. $\cosh x + \sinh x = e^x$
9. $\cosh x - \sinh x = e^{-x}$

Ex.: Find $\frac{dy}{dx}$ for the following functions:

1). $y = (x^2 + 1)\operatorname{sech} \ln x$

Sol.:

$$y = (x^2+1)\frac{2}{e^{\ln x} + e^{-\ln x}} = (x^2+1)\frac{2}{x + \frac{1}{x}} = \frac{2(x^2+1)}{\frac{x^2+1}{x}} = 2x$$

$$y = 2x \qquad , \qquad \frac{dy}{dx} = 2$$

2). $y = x\sinh 2x - \frac{1}{2}\cosh 2x$

Sol.:

$$\frac{dy}{dx} = x\cosh 2x.(2) + \sinh 2x - \frac{1}{2}\sinh 2x.(2) = 2x\cosh 2x$$

Ex.: Prove that:

1). $\frac{d}{dx}\tanh u = \text{sech}^2 u \frac{du}{dx}$

Sol.:

$$\frac{d}{dx}\tanh u = \frac{d}{dx}\left(\frac{\sinh u}{\cosh u}\right) = \frac{\cosh u \ \cosh u - \sinh u \ \sinh u}{\cosh^2 u}\frac{du}{dx}$$

$$= \frac{(\cosh^2 u \ - \sinh^2 u)}{\cosh^2 u}\frac{du}{dx}$$

Since $\cosh^2 u - \sinh^2 u = 1$ & $\text{sech}\, u = \frac{1}{\cosh u}$

$$\frac{d}{dx}\tanh u = \text{sech}^2 u \frac{du}{dx}$$

2). $\frac{d}{dx}\text{sech}\, u = -\text{sech}\, u \tanh u \frac{du}{dx}$

Sol.:

$$\frac{d}{dx}\text{sech}\, u = \frac{d}{dx}\frac{1}{\cosh u} = \frac{-\sinh u}{\cosh^2 u}\frac{du}{dx} = -\tanh u \,\text{sech}\, u \frac{du}{dx}$$

H. W.:

1). Simplify the following functions:

a). $\sinh(2\, lnx)$

b). $\cosh 3x - \sinh 3x$

c). $2\cosh(\ln x)$

2). Find $\frac{dy}{dx}$ of the following functions:

a). $y = 6\, sinh \frac{x}{3}$

b). $y = 2\sqrt{x}\tanh\sqrt{x}$

c). $y = \ln(\sinh x)$

The invers Hyperbolic Functions:

If u is any differentiable function of x then:

1. $\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
2. $\frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
3. $\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}$
4. $\frac{d}{dx}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}$
5. $\frac{d}{dx}\text{sech}^{-1}u = -\frac{1}{|u|\sqrt{1-u^2}}\frac{du}{dx}$
6. $\frac{d}{dx}\text{csch}^{-1}u = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$

Ex.: Find $\frac{dy}{dx}$ of the following functions:

1). $y = \cosh^{-1}(\sec x)$

Sol.:

$$\frac{dy}{dx} = \frac{1}{\sqrt{sec^2x - 1}}\sec x\ tan\ x = \frac{1}{\sqrt{tan^2x}}\sec x\ tan\ x = \sec x$$

$$1 + tan^2x = sec^2x$$

2). $y = \tanh^{-1}(\cos x)$

Sol.:

$$\frac{dy}{dx} = \frac{1}{1 - cos^2x}(-\sin x) = \frac{-\sin x}{sin^2x} = -\csc x$$

$$sin^2x + cos^2x = 1$$

3). $y = \coth^{-1}(\sec x)$

Sol.:

$$\frac{dy}{dx} = \frac{1}{1 - sec^2x}\sec x \tan x = \frac{\sec x \tan x}{-tan^2x} = -\csc x$$

4). $y = \text{sech}^{-1}(\sin 2x)$

Sol.:

$$\frac{dy}{dx} = \frac{1}{\sin 2x\ \sqrt{1-sin^2 2x}}2\cos 2x = \frac{2\cos 2x}{\sin 2x\ \sqrt{cos^2 2x}} = \frac{2\cos 2x}{\sin 2x\ \cos 2x}$$

$$= 2\csc 2x$$

Ex.: Prove that:

1). $\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

Sol.:

Let $y = \cosh^{-1} u \quad \Rightarrow \quad u = \cosh y$

$\frac{du}{dx} = \sinh y \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sinh y}\frac{du}{dx}$

Since $\cosh^2 y - \sinh^2 y = 1$

$\therefore \quad u^2 - \sinh^2 y = 1 \quad \Rightarrow \quad \sinh y = \sqrt{u^2 - 1}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \quad \Rightarrow \quad \frac{d}{dx}\cosh^{-1} y = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

2). $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}$

Sol.:

Let $y = \tanh^{-1} u \quad \Rightarrow \quad u = \tanh y$

$\frac{du}{dx} = \operatorname{sech}^2 y \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}\frac{du}{dx}$

Since $\operatorname{sech}^2 y + \tanh^2 y = 1$

$\therefore \quad \operatorname{sech}^2 y + u^2 = 1 \quad \Rightarrow \quad \operatorname{sech}^2 y = 1 - u^2$

$\therefore \quad \frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx} \quad \Rightarrow \quad \frac{d}{dx}\tanh^{-1} y = \frac{1}{1-u^2}\frac{du}{dx}$

Definite Integral

$\int_a^b f'(x).dx = f(x)\Big|_a^b = f(b) - f(a)$, a & b are Constant

Ex.:

$\int_1^2 3x^2.dx = 3\frac{x^3}{3}\Big|_1^2 = x^3\Big|_1^2 = (2)^3 - (1)^3 = 8 - 1 = 7$

Properties of Definite Integral:

1. $\int_a^a f(x).dx = 0$

   Ex.: $\int_2^2 x.dx = \frac{x^2}{2}\Big|_2^2 = 2 - 2 = 0$

2. $\int_a^b f(x).dx = -\int_b^a f(x).dx$ قلب الحدود

   Ex.: $\int_2^3 x^3.dx = -\int_3^2 x^3.dx$

3. $\int_a^b f(x).dx = \int_a^c f(x).dx + \int_c^b f(x).dx$ 1

4. $\int_a^b k.dx = k\,x\Big|_a^b = k\,(b-a)$, k Constant

   Ex.: $\int_3^{13} 4.dx = 4\,x\Big|_3^{13} = 4(13-3) = (4)(10) = 40$

5. $\int_a^b [f(x) \pm g(x)].dx = \int_a^b f(x).dx \pm \int_a^b g(x).dx$

Ex.: *If we have*

$\int_0^{16} f(x).dx = 12$ & $\int_{19}^{16} 12f(x).dx = 48$

Find $\int_0^{19} f(x).dx$

Sol.:

$$\int_0^{16} f(x).dx + (-)\int_{16}^{19} 12f(x).dx$$

$$-\int_{16}^{19} \frac{12}{12} f(x).dx = \frac{48}{12} = -\int_{16}^{19} f(x).dx = 4$$

$$\int_{16}^{19} f(x).dx = -4$$

$$\int_0^{16} f(x).dx + (-)\int_{16}^{19} 12f(x).dx = \int_0^{16} 12.dx + (-)\int_{16}^{19} (4).dx$$

$$= 12\,x \Big|_0^{16} - 4\,x \Big|_{16}^{19} = 12(16-0) - 4(19-16) = 192 - 12 = 180$$

Ex.: *If* $f(x) = \begin{cases} 2x & , \quad 0 \le x \le 2 \\ x^2 + 1 & , \quad 2 < x \le 3 \end{cases}$

Find $\int_0^3 f(x).\,dx$

Sol.:

$$\int_0^3 f(x).\,dx = \int_0^2 2x.\,dx + \int_2^3 (x^2 + 1).\,dx$$

$$= \frac{2x^2}{2}\bigg|_0^2 + \left(\frac{x^3}{3} + x\right)\bigg|_2^3 = (4 - 0) + \left[\left(\frac{27}{3} + 3\right) - \left(\frac{8}{3} + 2\right)\right]$$

$$= 4 + \left[\frac{27}{3} + 3 - \frac{8}{3} - 2\right] = 4 + \left[\frac{19}{3} + 1\right] = 4 + \frac{19 + 3}{3}$$

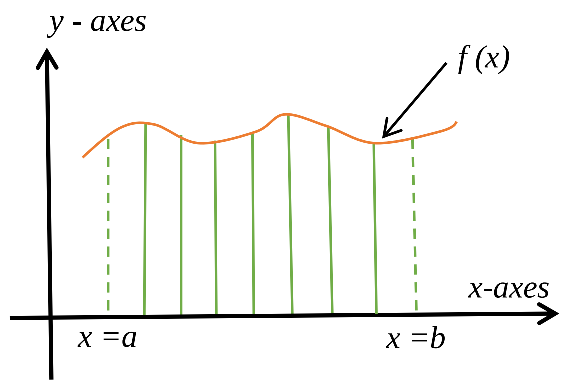
$$= 4 + \frac{22}{3} = \frac{12+22}{3} = \frac{34}{3}$$

Integral Applications

* من أهم التطبيقات في التكامل هو أيجاد المساحات، منها أيجاد المساحة تحت المنحني أو ما بين منحني وأحد المحاور $(x$ أو $y)$ أو ما بين منحنيين أو ما بين منحني وخط.

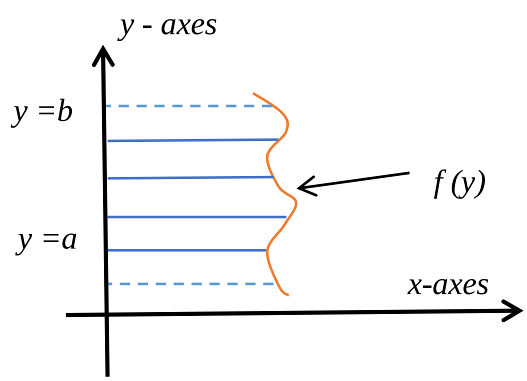
1. Area between curve and $x - axes$:

$$A = \int_{x=a}^{x=b} f(x).dx$$



2. Area between curve and $y - axes$:

$$A = \int_{y=a}^{y=b} f(y).dy$$



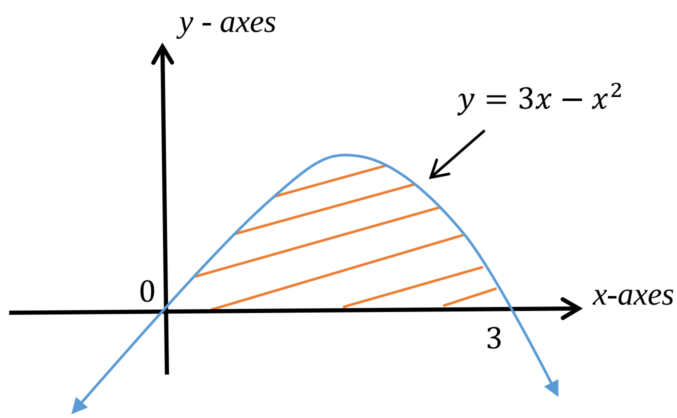
Ex.: *Find the area of the region bounded by* $y = 3x - x^2$ *and the* $x - axes$.

Sol.:

* لغرض حل أي سؤال، نتبع الخطوات التالية:

1. رسم الدالة. 2. تثبيت حدود التكامل. 3. حل التكامل.

x	$y = 3x - x^2$
-2	- 10
-1	- 4
0	0
1	2
2	2
3	0
4	- 4
5	- 10



$$y = 3x - x^2 = x(3 - x) \quad \rightarrow \quad x = 0 \quad , \quad x = 3$$

$$A = \int_0^3 (3x - x^2).\,dx = \left[3\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \left[3\frac{x^2}{2} - \frac{x^3}{3}\right] - \left[3\frac{(0)^2}{2} - \frac{(0)^3}{3}\right] = \left[\frac{27}{2} - \frac{27}{3}\right] - [0 - 0]$$

$$= 4.5 \ unit^2$$

Ex.: *Find the area between* $y = 2 - x^2$ *and the line* $y = -x$.

Sol.:

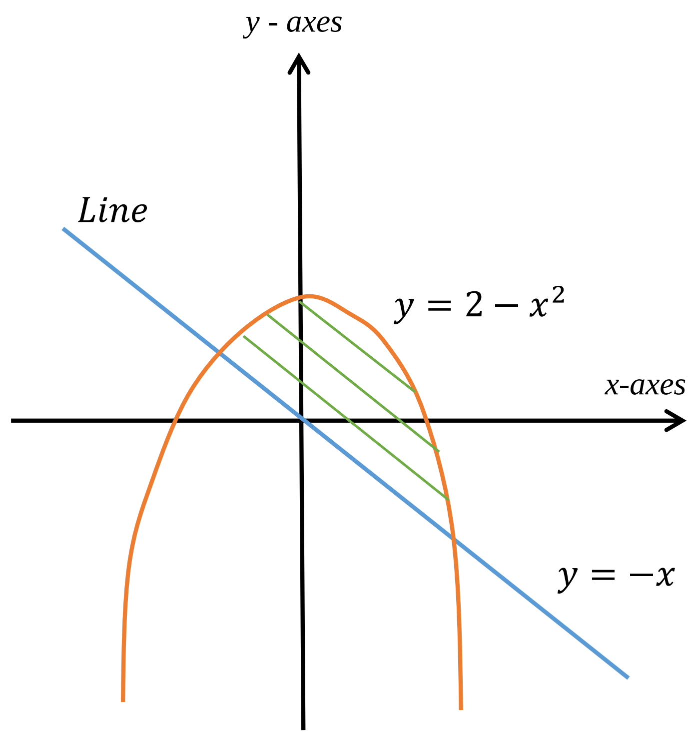
* عندما نرسم الشكل، نستفاد من الرسم للحصول على حدود التكامل (النقاط).

$y = 2 - x^2$

$$\begin{aligned} \text{If} \quad x = 0 &\Rightarrow y = 2 \\ x = 1 &\Rightarrow y = 1 \\ x = 2 &\Rightarrow y = -2 \\ x = 3 &\Rightarrow y = -7 \\ x = -1 &\Rightarrow y = 1 \\ x = -2 &\Rightarrow y = -2 \\ x = -3 &\Rightarrow y = -7 \end{aligned}$$

$y = -x$ *(Plot line)*

$$\begin{aligned} \text{If} \quad x = 0 &\Rightarrow y = 0 \\ x = 1 &\Rightarrow y = -1 \\ x = 2 &\Rightarrow y = -2 \\ x = 3 &\Rightarrow y = -3 \\ x = -1 &\Rightarrow y = 1 \\ x = -2 &\Rightarrow y = 2 \\ x = -3 &\Rightarrow y = 3 \end{aligned}$$



* هناك طريقة اخرى لإيجاد حدود التكامل، نساوي الدالتين:

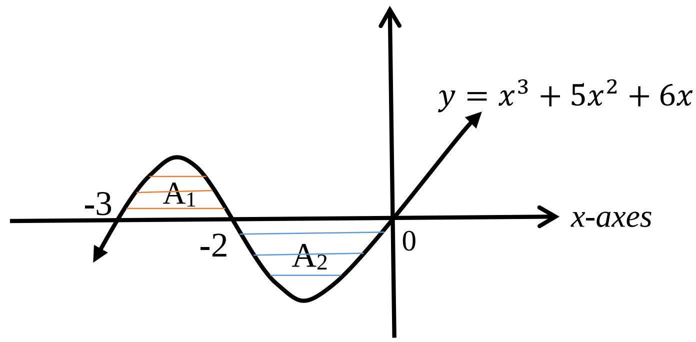
$$2 - x^2 = -x \quad \Rightarrow \quad x^2 - x - 2 = 0 \quad \Rightarrow \quad (x+1)(x-2) = 0$$

$$x = -1 = a \quad , \quad x = 2 = b$$

$$A = \int_a^b [y_1(x) - y_2(x)].dx = \int_{-1}^{2} [2 - x^2 + x]_{-1}^{2}.dx = \frac{9}{2}$$

$$= 4.5 \ \textit{unit area.}$$

Ex.: *Find the area of the region bounded by* $y = x^3 + 5x^2 + 6x$ *and the* $x-axes$.



Sol.:

$$y = x^3 + 5x^2 + 6x$$
$$= x(x^2 + 5x + 6)$$
$$= x(x+2)(x+3)$$

$x = 0 \quad , \quad x = -2 \quad , \quad x = -3$

$$A = A_1 + A_2$$

$$A_1 = \int_{-3}^{-2} (x^3 + 5x^2 + 6x).dx = \left[\frac{x^4}{4} + 5\frac{x^3}{3} + 6\frac{x^2}{2}\right]_{-3}^{-2}$$

$$= \left[\frac{(-2)^4}{4} + 5\frac{(-2)^3}{3} + 6\frac{(-2)^2}{2}\right] - \left[\frac{(-3)^4}{4} + 5\frac{(-3)^3}{3} + 6\frac{(-3)^2}{2}\right]$$

$$A_1 = \frac{5}{12} \ unit\ area.$$

$$|A_2| = \int_{-2}^{0} (x^3 + 5x^2 + 6x).dx = \left[\frac{x^4}{4} + 5\frac{x^3}{3} + 6\frac{x^2}{2}\right]_{-2}^{0}$$

$$= \left[\frac{(0)^4}{4} + 5\frac{(0)^3}{3} + 6\frac{(0)^2}{2}\right] - \left[\frac{(-2)^4}{4} + 5\frac{(-2)^3}{3} + 6\frac{(-2)^2}{2}\right]$$

$$|A_2| = \frac{8}{3} \ unit\ area.$$

$$A = A_1 + A_2 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} \ unit\ area.$$

H.W.: *Find the definite integrals for the following functions:*

1. $\int_0^1 \sqrt{x}\,.dx$

2. $\int_0^1 (x+x^2)\,.dx$

3. $\int_{-1}^3 \frac{1}{\sqrt{3x+4}}\,.dx$

4. $\int_1^2 (-2x+3)\,.dx$

5. $\int_{-1}^2 (x^2-2x)\,.dx$

6. $\int_1^3 \frac{1}{\sqrt{5x-1}}\,.dx$

7. $\int_4^9 \frac{t-8}{\sqrt{t}}\,.dt$

8. $\int_1^2 \left(\frac{4}{x}-\frac{2}{x^2}\right).dx$

9. $\int_0^1 \frac{x}{x+3}\,.dx$

10. $\int_0^2 \sqrt{x+3}\,.dx$

11. $\int_0^{\pi/3} (1-\sin\theta)\,.d\theta$

12. If $f(x)=\begin{cases}1+x & x<0\\ 2-x & 0\le x\le 2\\ \frac{3}{2} & x\ge 2\end{cases}$

    Find $\int_{-1}^3 f(x)\,.dx$

13. If $f(x)=\begin{cases}2x & 0\le x\le 1\\ 2 & 1<x\le 2\end{cases}$

    Find $\int_0^2 f(x)\,.dx$

14. If $f(x)=\begin{cases}1 & 0<x\le 1\\ -3 & 1<x\le 2\end{cases}$

    Find $\int_0^2 5f(x)\,.dx$

15. If $f(x)=\begin{cases}\frac{1}{x^2} & \frac{1}{2}\le x<1\\ \sqrt{x} & 1\le x\le 2\end{cases}$

    Find $\int_{\frac{1}{2}}^2 f(x)\,.dx$

16. If $f(x)=\begin{cases}1+x & 0\le x\le 1\\ -x+2 & 1<x\le 2\end{cases}$

    Find $\int_0^2 f(x)\,.dx$

17. If $f(x)=\begin{cases}x & 0\le x\le 1\\ x^2 & 1<x\le 2\end{cases}$

    Find $\int_0^2 f(x)\,.dx$

18. If $f(x)=\begin{cases}-x & 0\le x<1\\ -1+x & 1\le x\le 4\end{cases}$

    Find $\int_0^4 f(x)\,.dx$

19. If $f(x)=$

    $\begin{cases}x^2-1 & -2\le x\le 1\\ x-1 & 1<x\le 3\end{cases}$

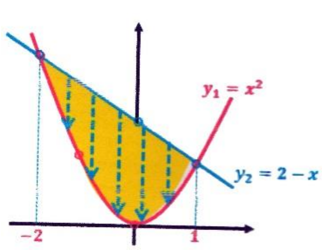
    Find $\int_{-2}^3 f(x)\,.dx$

20. If $f(x)=$

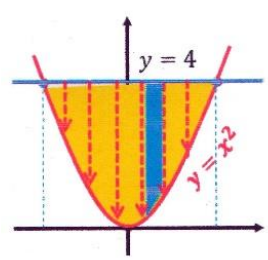
    $\begin{cases}\sqrt{x} & 1\le x\le 4\\ 2x^2-6x & 4<x\le 5\end{cases}$

    Find $\int_1^5 f(x)\,.dx$

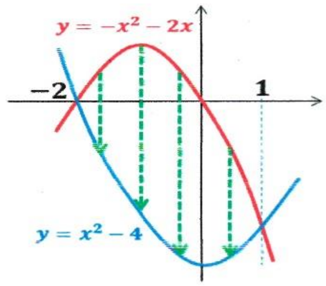
21. Find the area bounded by $y=x^2$ & $y=2-x$



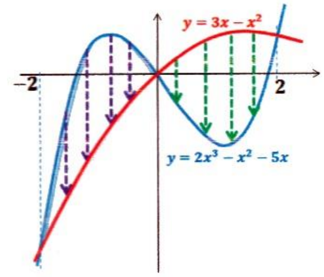
22. Find the area bounded by $y=x^2$ & $y=4$



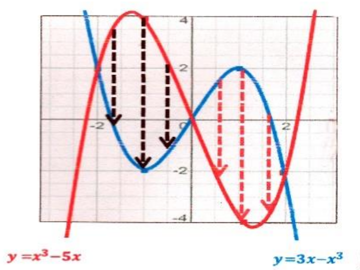
23. *Find the area bounded by* $y = -x^2 - 2x$ & $y = x^2$-4



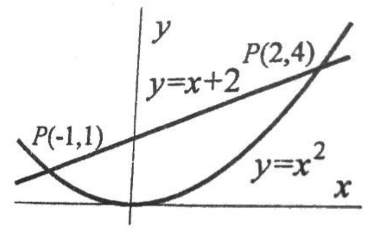
24. *Find the area bounded by* $y = 3x - x^2$ & $y = 2x^3 - x^2 - 5x$



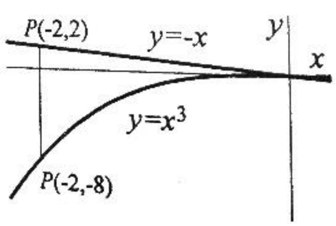
25. *Find the area bounded by* $y = x^3 - 5x$ & $y = 3x - x^3$



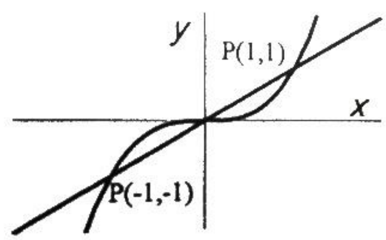
26. *Find the area bounded by* $y = x^2$ & $y = x + 2$



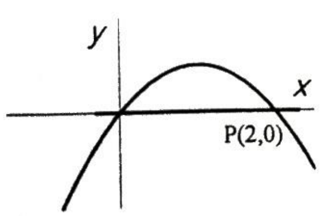
27. *Find the area bounded by* $y = x^3$ & $y = -x$ *in period* $[-2,0]$



28. *Find the area bounded by* $y = x^3$ & $y = x$



29. *Find the area bounded by* $y = 2x - x^2$ & $y = 0$



30. *Find the area bounded by* $y = 5x - x^3$ & $y = 0$

31. *Find the area bounded by* $y = x^3 - 3x$ & $y = x$

32. *Find the area bounded by* $y = x^3 - 6x$ & $y = -2x$

Indefinite Integral

التكامل غير المحدد: هو العملية العكسية لعملية الاشتقاق.
ملاحظة: في التكامل غير المحدد يجب ان نضيف C (integral constant) الى ناتج التكامل.

Properties of the indefinite integral

Rule. 1:

$$\int k\,dx = kx + c$$

$c = integral\ constant$
$k = number$

Ex.:

1. $\int 4\,dx = 4x + c$

2. $\int \pi\,dx = \pi x + c$

3. $\int \frac{dx}{2} = \int \frac{1}{2}dx = \frac{1}{2}x + c$

4. $\int \frac{1}{3}dm = \frac{1}{3}m + c$

Rule. 2:

$$\int \mathrm{x}^n\,dx = \frac{\mathrm{x}^{n+1}}{n+1} + \mathrm{c}$$

Ex.:

1. $\int x^2\,dx = \frac{1}{3}x^3 + c$

2. $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + c = \frac{-1}{x} + c$

3. $\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3}{4}x^{\frac{4}{3}} + c$

4. $\int \frac{1}{\sqrt{x^3}}\,dx = \int x^{\frac{-3}{2}}\,dx = \frac{x^{\frac{-1}{2}}}{\frac{-1}{2}} + c = \frac{-2}{\sqrt{x}} + c$

Rule. 3:

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)(a)} + c$$

a: تمثل مشتقة داخل القوس.

Ex.:

1. $\int (1-2x)^3\,dx = \frac{(1-2x)^4}{(4)(-2)} + c = \frac{-1}{8}(1-2x)^4 + c$

2. $\int \frac{1}{\sqrt[3]{x-1}}\,dx = \int (x-1)^{\frac{-1}{3}}dx = \frac{(x-1)^{\frac{2}{3}}}{\left(\frac{2}{3}\right)(1)} + c = \frac{3}{2}(x-1)^{\frac{2}{3}} + c$

Rule. 4:

$$\int K * f(x)\,dx = K\int f(x)\,dx$$

رقم مضروب في دالة

Ex.:

$$\int 2x^3\,dx = 2\int x^3dx = 2\frac{x^4}{4} + c = \frac{1}{2}x^4 + c$$

Rule. 5:

$$\int [f(x) \mp g(x)\,]dx = \int f(x)dx \mp \int g(x)dx$$

تستخدم هذه الصيغة في حالتي الجمع و الطرح.

Ex.:

1. $\int (x^2+6x-3)dx = \int x^2dx + \int 6x\,dx - \int 3dx$

$$= \frac{1}{3}x^3 + 3x^2 - 3x + c$$

2. $\int \left(x^6 + \frac{1}{x^3} - \sqrt[5]{x^2}\right)dx = \int \left(x^6 + x^{-3} - x^{\frac{2}{5}}\right)\,dx$

$$= \frac{x^7}{7} + \frac{x^{-2}}{-2} - \frac{x^{\frac{7}{5}}}{\frac{7}{5}} + c = \frac{1}{7}x^7 - \frac{1}{2x^2} - \frac{5}{7}x^{\frac{7}{5}} + c$$

Ex.:

* يجب تبسيط المعادلات (اذا كان هناك تبسيط) قبل اجراء التكامل.

1. $\int \frac{x^3 \cdot \sqrt{x}}{x}\,dx = \int x^2 \cdot x^{\frac{1}{2}}\,dx = \int x^{\frac{5}{2}}\,dx = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + c = \frac{2}{7}x^{\frac{7}{2}} + c$

2. $\int (1-x)(\sqrt{x})dx = \int \left(\sqrt{x} - x^{\frac{3}{2}}\right) dx = \int \left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\,dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + c$

3. $\int \frac{x^3+5x^2-4}{x^2}\,dx = \int \left(\frac{x^3}{x^2} + \frac{5x^2}{x^2} - \frac{4}{x^2}\right)\,dx = \int \left(x + 5 - \frac{4}{x^2}\right) dx$

$= \int (x + 5 - 4x^{-2})\,dx = \frac{x^2}{2} + 5x + 4\frac{1}{x} + c$

4. $\int \left(\frac{x^2-4}{x-2}\right) dx = \int \left(\frac{(x-2)(x+2)}{(x-2)}\right) dx = \int (x+2)\,dx = \frac{x^2}{2} + 2x + c$

5. $\int \frac{1}{x^2+6x+9}\,dx = \int \frac{1}{(x+3)(x+3)} dx = \int \frac{1}{(x+3)^2} dx = \int (x+3)^{-2} dx$

$= \frac{(x+3)^{-1}}{(-1)(1)} + c = \frac{-1}{x+3} + c$

بعض المتطابقات المهمة:

$$secx = \frac{1}{cosx} \quad . \quad cscx = \frac{1}{sinx} \quad . \quad tanx = \frac{sinx}{cosx}$$

$$cotx = \frac{1}{tanx} = \frac{cosx}{sinx}$$

$$sin^2(x) + cos^2(x) = 1 \quad , \quad tan^2(x) = sec^2(x) - 1$$

$$cot^2(x) = csc^2(x) - 1 \quad , \quad sin(2x) = 2sin(x)cos(x)$$

$$cos(2x) = cos^2(x) - sin^2(x) = 2cos^2(x) - 1 = 1 - 2sin^2(x)$$

Rule. 6:

$$\int sin(ax+b)\,dx = \frac{-cos(ax+b)}{a} + c$$

$$\int cos(ax+b)\,dx = \frac{sin(ax+b)}{a} + c$$

$$\int sec^2(ax+b)\,dx = \frac{tan(ax+b)}{a} + c$$

$$\int csc^2(ax+b)\,dx = \frac{-cot(ax+b)}{a} + c$$

$$\int sec(ax+b) \cdot tan(ax+b)dx = \frac{sec(ax+b)}{a} + c$$

$$\int csc(ax+b) \cdot cot(ax+b)dx = \frac{-csc(ax+b)}{a} + c$$

ملاحظة:

tan دائماً ترتبط بـ sec . و cot دائماً ترتبط بـ csc .

Ex.:

1. $\int (sinx - 3cosx)\, dx = -\, cosx - 3sinx\ + c$

2. $\int cos(3x-1)\, dx = \frac{sin(3x-1)}{3} + c = \frac{1}{3} sin(3x-1) + c$

3. $\int sec(2x) \cdot\ tan(2x) dx = \frac{sec(2x)}{2} + c = \frac{1}{2} sec(2x) + c$

4. $\int sin^2(x) + cos^2(x)\, dx = \int 1\, dx = x + c$

5. $\int \frac{secx}{cosx}\, dx = \int secx\, secx\, dx = \int sec^2 x\, dx = tanx + c$

6. $\int \frac{5}{sin^2 x}\, dx = \int 5\, csc^2 x\, dx = 5(-cotx) + c = -5cotx + c$

7. $\int cos(x) \cdot\ tan(x) dx = \int cosx \frac{sinx}{cosx} dx = \int sin(x) dx = -cosx + c$

8. $\int \frac{cos^3(x)-5}{1-sin^2(x)}\, dx = \int \frac{cos^3(x)-5}{cos^2(x)}\, dx = \int \left(\frac{cos^3(x)}{cos^2(x)} - \frac{5}{cos^2(x)} \right). dx$

$= \int \left(cos(x) - 5\, sec^2(x) \right). dx = sin(x) - 5tan(x) + c$

9. $\int tan^2(x) dx = \int (sec^2(x) - 1)\, dx = tan(x) - x + c$

10. $\int cot^2(x) dx = \int (csc^2(x) - 1)\, dx = -cot(x) - x + c$

11. $\int \frac{1}{1+sin(x)}\, dx = \int \frac{1}{1+sin(x)} \cdot \frac{1-sin(x)}{1-sin(x)} dx = \int \frac{1-sin(x)}{1-sin^2(x)} dx$

$$= \int \frac{1 - sin(x)}{cos^2(x)} dx = \int \frac{1}{cos^2(x)} - \frac{sin(x)}{cos^2(x)} dx$$

$$= \int (sec^2(x) - tanx \cdot secx)\, dx = tan(x) - sec(x) + c$$

- نضرب في مرافق المقام.

- عندما لا توجد حلول مباشرة، نلجأ الى المتطابقات في الحل.

Rule. 7:

$$\int (k)^{(ax+b)}\, dx = \frac{(k)^{(ax+b)}}{(a)\cdot \ln k} + c$$

$$\int e^{(ax+b)}\, dx = \frac{e^{(ax+b)}}{a\cdot \ln e} + c$$

a: تمثل مشتقة داخل القوس.

$$\ln e = 1$$

Ex.:

1. $\int (2)^{4x+5}\, dx = \frac{(2)^{4x+5}}{4\cdot \ln 2} + c$
2. $\int e^{x}\, dx = e^{x} + c$
3. $\int e^{(2-5x)}\, dx = \frac{e^{(2-5x)}}{(-5)} + c$
4. $\int 2^{x}\cdot e^{x}\, dx = \int (2e)^{x}\, dx = \frac{(2e)^{x}}{(1)\cdot \ln 2e} + c$

ملاحظة: عند وجود عملية ضرب بين رقمين لنفس الاس، فيمكن ضرب الاساس مرفوع للاس المشترك.

Rule. 8:

$$\int \frac{f'(x)}{f(x)}dx = ln|f(x)| + c$$

* عندما يكون البسط هو مشتقة المقام، فالتكامل هو ln المقام.

Ex.:

1. $\int x^{-1}dx = \int \frac{1}{x}dx = ln|x| + c$

2. $\int \frac{4}{x}dx = 4\int \frac{1}{x}dx = 4ln|x| + c$

3. $\int \frac{2}{x+2}dx = 2\int \frac{1}{x+2}dx = 2ln|x+2| + c$

4. $\int \frac{2x}{x^2+5}dx = ln|x^2+5| + c$

5. $\int \frac{x}{x^2+5}dx = \frac{1}{2}\int \frac{2x}{x^2+5}dx = \frac{1}{2}ln|x^2+5| + c$

6. $\int \frac{2}{4x+2}dx = \frac{1}{2}\int \frac{4}{4x+2}dx = \frac{1}{2}ln|4x+2| + c$

7. $\int tan(x)\,dx = \int \frac{sin(x)}{cos(x)}dx = -\int \frac{-sin(x)}{cos(x)}dx = -ln|cosx| + c$

$= ln|secx| + c$

8. $\int \cot(x)\,dx = \int \frac{1}{\tan(x)}dx = \int \frac{\cos(x)}{\sin(x)}dx = ln|\sin(x)| + c$

H.W.: *Find the indefinite integrals for the following functions:*

1. $\int \left(x - \frac{1}{cos^2 x}\right) \cdot dx$
2. $\int cos(2x) \cdot dx$
3. $\int e^{-3x} \cdot dx$
4. $\int 2x \cdot dx$
5. $\int 3\sqrt{x} \cdot dx$
6. $\int (x-2)^2 \cdot dx$
7. $\int \left(x + \frac{1}{x}\right) \cdot dx$
8. $\int \left(\frac{1}{x^7} + \frac{1}{x^2}\right) \cdot dx$
9. $\int 10 \cdot dx$
10. $\int (x^2 + cos\, x) \cdot dx$
11. $\int sec^2(x-1) \cdot dx$
12. $\int sin(4x) \cdot dx$
13. $\int \left(x^{3/2} + x^{3/5}\right) \cdot dx$
14. $\int \frac{1}{\sqrt{x+1}} \cdot dx$
15. $\int \left(\sqrt{t} - sec^2 t + 1\right) \cdot dt$
16. $\int tan^2 x \cdot dx$
17. $\int \left(\frac{1}{csc\, x \cdot cot\, x \cdot cosx}\right) dx$
18. $\int \left(x^{3/2} + x^{5/2}\right) \cdot dx$
19. $\int \frac{x^2 - 2x + 1}{\sqrt{x}} \cdot dx$
20. $\int \left(y + \frac{1}{y}\right)^2 \cdot dy$
21. $\int \left(x - \frac{1}{\sqrt{x}}\right)^3 \cdot dx$
22. $\int \left(\frac{4}{x^3} + \frac{5}{x^2} - \frac{7}{x}\right) \cdot dx$
23. $\int \left(\sqrt{x} + 2csc^2 x\right) \cdot dx$
24. $\int \left[x^{-1}\left(\sqrt{x} + 5x\right)\right] \cdot dx$
25. $\int \left[x^{\frac{-1}{3}}\left(\sqrt{x} + \frac{5}{x}\right)\right] \cdot dx$
26. $\int \left(\sqrt{x+1} - \sqrt{x+2}\right) \cdot dx$
27. $\int cot^2 x \cdot dx$
28. $\int (e^x + e^{-x})^2 \cdot dx$
29. $\int \left(\frac{(x+1)^2 - 1}{(x+1)^3}\right) \cdot dx$
30. $\int \frac{(1 + secx)cotx}{cscx} \cdot dx$
31. $\int \frac{7^{2x} - e^{-5x}}{3} \cdot dx$
32. $\int \left(\frac{4^{2x} - e^{-5x}}{4}\right) \cdot dx$
33. $\int \sqrt{2x - x^2} \cdot dx$
34. $\int \frac{x^2 + 2}{1 + x^2} \cdot dx$
35. $\int \frac{\cot x}{\sqrt{1 - cos^2 x}} \cdot dx$

Integration by Parts التكامل بالتجزئة

هي أحدى الطرق المهمة لاجراء التكاملات خصوصاً التي تجمع بين نوعين مختلفين من الدوال.

وتتلخص هذه الطريقة عندما يكون هناك اقتران لدالتين وليس هناك مشتقة لأي منهما.

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$\int \ln x \, dx \Rightarrow \begin{cases} u = \ln x \overset{\acute{u}}{\Rightarrow} du = \frac{1}{x} dx \\ dv = dx \xrightarrow{\int \square} v = x \end{cases}$	$\int \sin^{-1} x \, dx \Rightarrow \begin{cases} u = \sin^{-1} x \overset{\acute{u}}{\Rightarrow} du = \frac{dx}{\sqrt{1-x^2}} \\ dv = dx \xrightarrow{\int \square} v = x \end{cases}$
$\int x^n \ln x \, dx \Rightarrow \begin{cases} u = \ln x \overset{\acute{u}}{\Rightarrow} du = \frac{1}{x} dx \\ dv = x^n dx \xrightarrow{\int \square} v = \frac{x^{(n+1)}}{n+1} \end{cases}$	$\int x^n \tan^{-1} x \, dx \Rightarrow \begin{cases} u = \tan^{-1} x \overset{\acute{u}}{\Rightarrow} du = \frac{dx}{1+x^2} \\ dv = x^n dx \xrightarrow{\int \square} v = \frac{x^{(n+1)}}{n+1} \end{cases}$
$\int x^n e^x dx \Rightarrow \begin{cases} u = x^n \overset{\acute{u}}{\Rightarrow} du = nx^{(n-1)dx} \\ dv = e^x \, dx \xrightarrow{\int \square} v = e^x \end{cases}$	$\int x^n \sin x \, dx \Rightarrow \begin{cases} u = x^n \overset{\acute{u}}{\Rightarrow} du = nx^{(n-1)dx} \\ dv = sinx \, dx \xrightarrow{\int \square} v = -\cos x \end{cases}$

Ex.:

1. $\int \ln x \, . \, dx$, $\int \ln x \, . \, dx \Rightarrow \begin{cases} u = \ln x \overset{\acute{u}}{\Rightarrow} du = \frac{1}{x} . dx \\ dv = dx \xrightarrow{\int} v = x \end{cases}$

$$u = \ln x \quad , \quad du = \frac{1}{x} . dx$$

$$dv = dx \quad , \quad v = x$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int \ln x \, . \, dx = x \, . \ln x - \int x . \frac{1}{x} . dx = x . \ln x - x + c$$

2. $\int x.sinx.dx$

$u = x \quad , \quad du = dx$

$dv = sinx.dx \quad , \quad v = -cosx$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int x.sinx.dx = -x.cosx - \int -cosx.dx$$

$$= -x.cosx + \int cosx.dx = -x.cosx + sinx + c$$

3. $\int x^2.cosx\, dx$

$u = x^2 \quad , \quad du = 2x.dx$

$dv = cosx.dx \quad , \quad v = sinx$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int x^2.cosx\, dx = x^2 sinx - \int 2xsinx\, dx$$

$$= x^2.sinx - 2\int x.sinx.dx = x^2.sinx - 2[-x.cosx - \int -cosx.dx]$$

$u = x \quad , \quad du = dx$

$dv = sinx.dx \quad , \quad v = -cosx$

$\int x^2.cosx\, dx = x^2.sinx - 2[-x.cosx + sinx] + c$

4. $\int x^5 . cosx . dx$

$$u = x^5 \quad , \quad dv = cosx\ dx$$

في هذا السؤال يكون الحل صعب ويجب تكرار الحل بالتجزئة خمس مرات. لذلك يتم الحل باستخدام Tabular method.

u		dv
x^5	+	$cosx$
$5x^4$	-	$sinx$
$20x^3$	+	$-cosx$
$60x^2$	-	$-sinx$
$120x$	+	$cosx$
120	-	$sinx$
0		$-cosx$

$$\int x^5 . cosx\ dx = x^5 sinx + 5x^4 cosx - 20x^3 sinx - 60x^2 cosx + 120x sinx$$
$$+120 cosx + c.$$

5. $\int x^2 . \ln x\ \ dx$

$$u = \ln x \quad , \quad du = \frac{1}{x} . dx$$

$$dv = x^2 . dx \quad , \quad v = \frac{x^3}{3}$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int x^2 . \ln x . dx = \ln x . \frac{x^3}{3} - \int \frac{x^3}{3} . \frac{1}{x} . dx = \ln x . \frac{x^3}{3} - \int \frac{x^2}{3} . dx$$

$$= \ln x . \frac{x^3}{3} - \frac{1}{9} x^3 + c$$

6. $\int x.e^x.dx$

$u = x \quad , \quad du = dx$

$dv = e^x.dx \quad , \quad v = e^x$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int x.e^x.dx = x.e^x - \int e^x.dx = x.e^x - e^x + c$$

7. $\int \tan^{-1} x\,.dx$

$u = \tan^{-1} x \quad , \quad du = \frac{1}{1+x^2}.dx$

$dv = dx \quad , \quad v = x$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int \tan^{-1} x\,.dx = x\,.\tan^{-1} x - \int x\,\frac{1}{1+x^2}.dx = x\,.\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}.dx$$

$$= x\,.\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

8. $\int sinx.e^x.dx$

$u = sinx \quad , \quad du = cosx.dx$

$dv = e^x.dx \quad , \quad v = e^x$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int sinx.e^x.dx = sinx.e^x - \int e^x.cosx.dx$$

$u = cosx \quad , \quad du = -sinx.dx$

$dv = e^x.dx \quad , \quad v = e^x$

$$\int sinx.e^x\,.dx = sinx.e^x - \left[e^x.cosx - \int e^x\,(-sinx).dx\right]$$

$$\therefore \int sinx.e^x.dx = sinx.e^x - cosx.e^x - \int e^x.sinx.dx$$

$$2\int sinx.e^x.dx = sinx.e^x - cosx.e^x = \int sinx.e^x.dx = \frac{sinx.e^x - cosx.e^x}{2} + C$$

9. $\int cos\left(\sqrt{2x+1}\right)dx$

Let $Z=\sqrt{2x+1}\ \Rightarrow\ Z^2=2x+1\ \Rightarrow\ 2z.dz=2.dx\ \Rightarrow\ dx=z.dz$

$\int cos\left(\sqrt{2x+1}\right)dx=\int z.cos(z).dz$

$u=z \qquad , \qquad du=dz$

$dv=cos(z).dz \qquad , \qquad v=sin(z)$

$$\int cos\left(\sqrt{2x+1}\right)dx=z.sin(z)-\int sin(z).dz=z.sin(z)+cos(z)+c$$

$$=\left(\sqrt{2x+1}\right)sin\left(\sqrt{2x+1}\right)+cos\left(\sqrt{2x+1}\right)+c$$

10. $\int x^3.e^{x^2}.dx$

$z=x^2 \qquad , \qquad dz=2x.dx \qquad , \qquad dx=\frac{dz}{2x}$

$$\int x^3.e^{x^2}.dx=\int x^3.e^z.\frac{dz}{2x}=\frac{1}{2}\int x^2.e^z.dz=\frac{1}{2}\int z.e^z.dz$$

$u=z \qquad , \qquad du=dz$

$dv=e^z.dz \qquad , \qquad v=e^z$

$$\int x^3.e^{x^2}.dx=\frac{1}{2}\int z.e^z.dz=\frac{1}{2}\left[z.e^z-\int e^z.dz\right]=\frac{1}{2}[z.e^z-e^z]+c$$

$$=\frac{1}{2}\left[x^2.e^{x^2}-e^{x^2}\right]+c$$

11. $\int e^{\sqrt{x}}\,.dx$

Let $z = \sqrt{x}$, $z^2 = x$, $2z.dz = dx$

$$\int e^{\sqrt{x}}\,.dx = \int e^{Z}\,.2z.dz = 2\int Z.e^{Z}.dz$$

$u = z$, $du = dz$

$dv = e^{z}.dz$, $v = e^{z}$

$$\int e^{\sqrt{x}}\,.dx = 2\int Z.e^{Z}.dz = 2\left[z.e^{z} - \int e^{z}\,.dz\right] = 2[z.e^{z} - e^{z}] + c$$

$$= 2\left[\sqrt{x}.e^{\sqrt{x}} - e^{\sqrt{x}}\right] + c$$

12. $\int sin(2x).e^{cos(x)}\,dx$

Let $z = cos(x)$, $dz = -sin(x).dx$, $dx = \dfrac{dz}{-sin(x)}$

$$\int sin(2x).e^{cosx}\,dx = \int sin(2x).e^{z}\,\frac{dz}{-sin(x)} = \int 2sin(x).cos(x).e^{z}\,\frac{dz}{-sin(x)}$$

$$= -2\int z\,.e^{z}.dz$$

$sin(2x) = 2\sin(x)\cdot\cos(x)$

$u = z \overset{\acute{u}}{\Rightarrow} du = dz$

$dv = e^{z}\cdot dz \overset{\int}{\Longrightarrow} v = e^{z}$

$$\int u\cdot dv = u\cdot v - \int v\cdot du$$

$$\int sin(2x)\cdot e^{cos(x)}\,dx = -2\int z\cdot e^{z}\cdot dz = -2z\cdot e^{z} + 2\int e^{z}\cdot dz$$

$$= -2z\cdot e^{z} + 2e^{z} + c = -2cos(x)\cdot e^{cosx} + 2e^{cos(x)} + c$$

13. $\int \frac{e^x}{csc(x)} . dx$

$$\int \frac{e^x}{csc(x)}\, dx = \int e^x \cdot sin(x) \cdot \, dx$$

$$u = e^x \quad \overset{\acute{u}}{\Rightarrow} \quad du = e^x \cdot dx$$

$$dv = sin(x) \cdot dx \quad \overset{\int}{\Longrightarrow} \quad v = -cos(x)$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int \frac{e^x}{csc(x)}\, dx = \int e^x . sin(x) . dx = -e^x . cos(x) + \int e^x . cos(x) . dx$$

$$u = e^x \quad \overset{\acute{u}}{\Rightarrow} \quad du = e^x . dx$$

$$dv = cos(x) \cdot dx \quad \overset{\int}{\Longrightarrow} \quad v = sin(x)$$

$$\int e^x \cdot sin(x) \cdot \, dx = -e^x . cos(x) + \int e^x . cos(x) . dx$$

$$\int e^x \cdot sin(x) \cdot \, dx = -e^x \cdot cos(x) + \left[e^x \cdot sin(x) - \int sin(x) \cdot e^x \cdot dx \right]$$

$$\int e^x \cdot sin(x) \cdot \, dx = -e^x \cdot cos(x) + e^x \cdot sin(x) - \int e^x \cdot sin(x) \cdot dx$$

$$\int e^x \cdot sin(x) \cdot \, dx + \int e^x \cdot sin(x) \cdot dx = -e^x \cdot cos(x) + e^x \cdot sin(x)$$

$$2 \int e^x \cdot sin(x) \cdot dx = -e^x \cdot cos(x) + e^x \cdot sin(x)$$

$$\int e^x \cdot sin(x) \cdot dx = \frac{-e^x \cdot cos(x) + e^x \cdot sin(x)}{2} + C$$

H.W.:

1. $\int x \cdot e^{-x} \cdot dx$

2. $\int x \cdot \sin(2x) \cdot dx$

3. $\int \frac{\ln x}{\sqrt{x}} \cdot dx$

4. $\int e^{x} \cdot \cos(2x) \cdot dx$

Integration by substitution التكامل بالتعويض

التكامل بطريقة التعويض هو أحدى الطرق للحل وذلك بفرض (z) لمشتقة موجودة في تكاملات دوال حاصل ضرب أو قسمة يصعب حلها بقواعد التكامل الاساسية.

الحالات العامة لطرق الحل بالتعويض

1. الاقتران المرفوع لقوة للاقواس

Ex.:

1. $\int 5x^4 \cdot (x^5+2)^6 \cdot dx$

خطوات الحل:

1. نفرض (x^5+2) تساوي Z.
2. نشتق $(x^5+2)^6$ بدون الأس.
3. نشتق Z.

Let $Z = x^5 + 2 \quad \Rightarrow \quad dz = 5x^4 \cdot dx \quad \Rightarrow \quad dx = \frac{dz}{5x^4}$

$$\int 5x^4 \cdot (x^5+2)^6 \cdot dx = \int 5x^4 \cdot (Z)^6 \cdot \frac{dz}{5x^4} = \int Z^6 \cdot dz = \frac{Z^7}{7} + C = \frac{(x^5+2)^7}{7} + C$$

2. $\int 2x \cdot (x^2+1)^7 \cdot dx$

Let $Z = x^2 + 1 \quad \Rightarrow \quad dz = 2x \cdot dx \quad \Rightarrow \quad dx = \frac{dz}{2x}$

$$\int 2x \cdot (x^2+1)^7 \cdot dx = \int 2x \cdot (Z)^7 \cdot \frac{dz}{2x} = \int Z^7 \cdot dz = \frac{Z^8}{8} + C = \frac{(x^2+1)^8}{8} + C$$

2. اذا كان الاقتران داخل الجذر او الجذر نفسه

Ex.

1. $\int x \cdot \sqrt{x^2+1} \cdot dx$

Let $Z = \sqrt{x^2+1} \Rightarrow Z^2 = x^2+1 \Rightarrow 2zdz = 2x \cdot dx \Rightarrow dx = \frac{2zdz}{2x} = \frac{zdz}{x}$

$$\int x \cdot \sqrt{x^2+1} \cdot dx = \int x \cdot z \cdot \frac{zdz}{x} = \int Z^2 \cdot dz = \frac{Z^3}{3} + C = \frac{\left(\sqrt{x^2+1}\right)^3}{3} + C$$

2. $\int x^3 \cdot \sqrt{x^2+1} \cdot dx$

Let $Z = \sqrt{x^2+1} \Rightarrow Z^2 = x^2+1 \Rightarrow 2zdz = 2x \cdot dx \Rightarrow dx = \frac{2zdz}{2x} = \frac{zdz}{x}$

$$\int x^3 \cdot \sqrt{x^2+1} \cdot dx = \int x^3 \cdot z \cdot \frac{zdz}{x} = \int x^2 \cdot z^2 \cdot dz$$

يجب التخلص من x

$Z^2 = x^2 + 1 \Rightarrow x^2 = z^2 - 1$

$$\int (Z^2-1) \cdot Z^2 \cdot dz = \int (Z^4 - Z^2) \cdot dz = \frac{Z^5}{5} - \frac{Z^3}{3} + C$$

$$= \frac{\left(\sqrt{x^2+1}\right)^5}{5} - \frac{\left(\sqrt{x^2+1}\right)^3}{3} + C$$

3. $\int_0^4 x \cdot \sqrt{x^2+9} \cdot dx$

Let $Z = \sqrt{x^2+9} \Rightarrow Z^2 = x^2+9 \Rightarrow 2z \cdot dz = 2x \cdot dx \Rightarrow dx = \frac{2zdz}{2x} = \frac{zdz}{x}$

نقوم بتغيير حدود التكامل

if $x = 0 \Rightarrow Z = \sqrt{(0)^2+9} = \sqrt{9} = 3$

if $x = 4 \Rightarrow Z = \sqrt{(4)^2+9} = \sqrt{25} = 5$

$$\int_3^5 x \cdot z \cdot \frac{zdz}{x} = \int_3^5 Z^2 \cdot dz = \frac{Z^3}{3}\Big]_3^5 = \frac{125}{3} - \frac{27}{3} = \frac{98}{3}$$

3. اذا كانت زاوية الاقتران الدائري غير خطية

Ex.:

1. $\int x^3 \cdot sin(2x^4) \cdot dx$

Let $Z = 2x^4 \Rightarrow dz = 8x^3 \cdot dx \Rightarrow dx = \frac{dz}{8x^3}$

$$\int x^3 \cdot sin(2x^4) \cdot dx = \int x^3 \cdot sin(z) \cdot \frac{dz}{8x^3} = \frac{1}{8}\int sin(z) \cdot dz = \frac{1}{8}(-cos(z)) + C$$

$$= \frac{-1}{8} cos(2x^4) + C$$

2. $\int 3x^2 \cdot cos(x^3 + 1) \cdot dx$

Let $Z = x^3 + 1 \Rightarrow dz = 3x^2 \cdot dx \Rightarrow dx = \frac{dz}{3x^2}$

$$\int 3x^2 \cdot cos(x^3 + 1) \cdot dx = \int 3x^2 \cdot cos(z) \cdot \frac{dz}{3x^2} = \int cos(z) \cdot dz = sin(z) + C$$

$$= sin(x^3 + 1) + C$$

4. اذا كانت قوة الاقتران الأسي غير خطية

Ex.:

1. $\int cosx \cdot e^{(sinx)} \cdot dx$

Let $Z = sinx \Rightarrow dz = cosx \cdot dx \Rightarrow dx = \frac{dz}{cosx}$

$$\int cosx \cdot e^{(sinx)} \cdot dx = \int cosx \cdot e^{(z)} \cdot \frac{dz}{cosx} = \int e^{(z)} \cdot dz = e^{(z)} + c = e^{(sinx)} + c$$

2. $\int \frac{(ln\,x)^3}{x} \cdot dx$

Let $Z = ln\,x \Rightarrow dz = \frac{1}{x} \cdot dx \Rightarrow dx = x \cdot dz$

$$\int \frac{(ln\,x)^3}{x} \cdot dx = \int \frac{(z)^3}{x} \cdot x \cdot dz = \int (z)^3 \cdot dz = \frac{z^4}{4} + c = \frac{(ln\,x)^4}{4} + c$$

H.W.:

1. $\int sec^2(4x^3)\cdot x^2 dx$

2. $\int (1+x^7)^{100}x^6 dx$

3. $\int \left(\frac{2x^3+3x^2}{x^4+2x^3}\right)\cdot dx$

4. $\int tanx\, dx$

5. $\int \frac{sin(2x)}{\sqrt{1+sinx}}\cdot dx$

6. $\int \frac{x}{cos^2(x^2)}\cdot dx$

7. $\int x\sqrt{1+2x^2}\cdot dx$

8. $\int xe^{x^2}\cdot dx$

9. $\int sec^2(4x+2)dx$

10. $\int cos^3x\cdot sinx\cdot dx$

11. $\int \frac{x}{\sqrt{4-9x^2}}\cdot dx$

12. $\int \frac{x^2}{\sqrt{x^3+1}}\cdot dx$

13. $\int sin(sinx)cosx\, dx$

14. $\int x.cos(x^2)\cdot dx$

15. $\int \frac{sec^2(\sqrt{x})}{\sqrt{x}}\cdot dx$

16. $\int \frac{x}{\sqrt[3]{1-2x^2}}\cdot dx$

17. $\int x^2sec^2(x^3)\cdot dx$

18. $\int (2-x^4)^3\, x^3\cdot dx$